MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 7 | **MATH 2050** | WINTER 2018

SOLUTIONS

[3] 1. (a) (i) $M = \begin{bmatrix} 15 & 3 & -6 \\ -10 & -2 & 4 \\ -65 & -13 & 26 \end{bmatrix}$

[1] (ii) $C = \begin{bmatrix} 15 & -3 & -6 \\ 10 & -2 & -4 \\ -65 & 13 & 26 \end{bmatrix}$

[3] (iii) $AC^T = \begin{bmatrix} 15 & 3 & -6 \\ -10 & -2 & 4 \\ -65 & -13 & 26 \end{bmatrix} \begin{bmatrix} 15 & 10 & -65 \\ -3 & -2 & 13 \\ -6 & -4 & 26 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I$ so $\det(A) = 0$.

[1] (iv) Since $\det(A) = 0$, A is not invertible.

[3] (b) (i) $M = \begin{bmatrix} -12 & 21 & 15 \\ -9 & 17 & 10 \\ 21 & -48 & -30 \end{bmatrix}$

[1] (ii) $C = \begin{bmatrix} -12 & -21 & 15 \\ 9 & 17 & -10 \\ 21 & 48 & -30 \end{bmatrix}$

[3] (iii) $AC^T = \begin{bmatrix} -12 & 21 & 15 \\ -9 & 17 & 10 \\ 21 & -48 & -30 \end{bmatrix} \begin{bmatrix} -12 & 9 & 21 \\ -21 & 17 & 48 \\ 15 & -10 & -30 \end{bmatrix} = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix} = 15I$ so $\det(A) = 15$.

[2] (iv) $A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{15} \begin{bmatrix} -12 & -9 & 21 \\ 21 & 17 & -48 \\ 15 & 10 & -30 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} & \frac{7}{5} \\ -\frac{7}{5} & \frac{17}{15} & \frac{16}{5} \\ 1 & -\frac{2}{3} & -2 \end{bmatrix}$

[5] 2. (a) We expand along the third row:

$$\det(A) = 4 \begin{vmatrix} 7 & 6 \\ 5 & 4 \end{vmatrix} - \begin{vmatrix} 3 & 6 \\ 9 & 4 \end{vmatrix} + 0 = 4(-2) - (-42) = 34.$$

(Note that we could also expand along the third column.)

[9] (b) We expand along the first row:

$$\det(B) = 0 - 2 \begin{vmatrix} -1 & -1 & -2 \\ -5 & 3 & -3 \\ 4 & 0 & -3 \end{vmatrix} + (-3) \begin{vmatrix} -1 & 2 & -2 \\ -5 & 0 & -3 \\ 4 & -1 & -3 \end{vmatrix} - 0.$$

To compute the first of these 3×3 determinants, we expand along the third row (or the second column):

$$\begin{vmatrix} -1 & -1 & -2 \\ -5 & 3 & -3 \\ 4 & 0 & -3 \end{vmatrix} = 4 \begin{vmatrix} -1 & -2 \\ 3 & -3 \end{vmatrix} - 0 + (-3) \begin{vmatrix} -1 & -1 \\ -5 & 3 \end{vmatrix} = 4(9) - 3(-8) = 60.$$

To compute the second 3×3 determinant, we expand along the second row (or the second column):

$$\begin{vmatrix} -1 & 2 & -2 \\ -5 & 0 & -3 \\ 4 & -1 & -3 \end{vmatrix} = -(-5)\begin{vmatrix} 2 & -2 \\ -1 & -3 \end{vmatrix} + 0 - (-3)\begin{vmatrix} -1 & 2 \\ 4 & -1 \end{vmatrix} = 5(-8) + 3(-7) = -61.$$

Thus

$$\det(B) = -2(60) - 3(-61) = 63.$$

[9] (c) We expand along the fourth column:

$$\det(C) = 0 + (-2)\begin{vmatrix} 1 & 2 & -3 \\ 7 & 1 & 1 \\ -3 & -4 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 2 & -3 \\ 1 & 4 & 5 \\ -3 & -4 & 2 \end{vmatrix} + 0.$$

For the two 3×3 determinants, we have no obvious row or column along which to expand, so we'll choose the first row in each case. First, we have

$$\begin{vmatrix} 1 & 2 & -3 \\ 7 & 1 & 1 \\ -3 & -4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix} - 2\begin{vmatrix} 7 & 1 \\ -3 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 7 & 1 \\ -3 & -4 \end{vmatrix} = 6 - 2(17) - 3(-25) = 47.$$

Next,

$$\begin{vmatrix} 1 & 2 & -3 \\ 1 & 4 & 5 \\ -3 & -4 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ -4 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 5 \\ -3 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 4 \\ -3 & -4 \end{vmatrix} = 28 - 2(17) - 3(8) = -30.$$

Thus

$$\det(C) = -2(47) - (-30) = -64.$$

[7] 3. (a) We first row-reduce A to row-echelon form using only the third elementary row operation:

$$A = \begin{bmatrix} 3 & -6 & 2 \\ -3 & 0 & -1 \\ 1 & -1 & 4 \end{bmatrix} \xrightarrow[R_3\to R_3-\left(\frac{1}{3}\right)R_1]{R_2\to R_2-(-1)R_1} \begin{bmatrix} 3 & -6 & 2 \\ 0 & -6 & 1 \\ 0 & 1 & \frac{10}{3} \end{bmatrix}$$

$$\xrightarrow{R_3\to R_3-\left(-\frac{1}{6}\right)R_2} \begin{bmatrix} 3 & -6 & 2 \\ 0 & -6 & 1 \\ 0 & 0 & \frac{7}{2} \end{bmatrix} = U.$$

Now we compose the matrix L. We subtracted -1 times the 1st row from the 2nd row, so the $(2,1)$ element is -1. We subtracted $\frac{1}{3}$ times the 1st row from the 3rd row, so the $(3,1)$ element is $\frac{1}{3}$. We subtracted $-\frac{1}{6}$ times the 2nd row from the 3rd row, so the $(3,2)$ element is $-\frac{1}{6}$. Hence

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{3} & -\frac{1}{6} & 1 \end{bmatrix}.$$

So now we want to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 13 \\ -8 \\ -21 \end{bmatrix}$. This is equivalent to $LU\mathbf{x} = \mathbf{b}$, and so first we solve $L\mathbf{y} = \mathbf{b}$ by forward-substitution. We have

$$\begin{aligned} y_1 &= 13 \\ y_2 &= -8 + y_1 = 5, \\ y_3 &= -21 + \frac{1}{6}y_2 - \frac{1}{3}y_1 = -21 + \frac{5}{6} - \frac{13}{3} = -\frac{49}{2}. \end{aligned}$$

Now we use back-substitution to solve $U\mathbf{x} = \mathbf{y}$ and so

$$\begin{aligned} x_3 &= \frac{2}{7}\left(-\frac{49}{2}\right) = -7 \\ x_2 &= -\frac{1}{6}(5 - x_3) = -2 \\ x_1 &= \frac{1}{3}(13 - 2x_3 + 6x_2) = 5. \end{aligned}$$

Hence the solution is

$$\mathbf{x} = \begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}.$$

[3] (b) Again, we can try to row-reduce B to row-echelon form using only the third elementary row operation:

$$A = \begin{bmatrix} 3 & -6 & 2 \\ -3 & 6 & -1 \\ 1 & -1 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - \left(\frac{1}{3}\right)R_1]{R_2 \to R_2 - (-1)R_1} \begin{bmatrix} 3 & -6 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & \frac{10}{3} \end{bmatrix}.$$

However, there is now no way to proceed without interchanging the second and third rows (via the first elementary row operation). As such, no LU factorisation is possible. (These situations require the introduction of a third matrix called a permutation matrix, giving rise to a PLU factorisation. If you're interested, the Goodaire textbook provides more information about this kind of factorisation.)