MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 6 **MATH 2050** Winter 2018

SOLUTIONS

[6] 1. (a) Using Gaussian elimination, we have

$$\begin{bmatrix} 2 & 3 & -4 \\ 4 & 1 & 9 \\ 0 & 5 & -17 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{3}{2} & -2 \\ 4 & 1 & 9 \\ 0 & 5 & -17 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & \frac{3}{2} & -2 \\ 0 & -5 & 17 \\ 0 & 5 & -17 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{5}R_2} \begin{bmatrix} 1 & \frac{3}{2} & -2 \\ 0 & 1 & -\frac{17}{5} \\ 0 & 5 & -17 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 5R_2} \begin{bmatrix} 1 & \frac{3}{2} & -2 \\ 0 & 1 & -\frac{17}{5} \\ 0 & 0 & 0 \end{bmatrix}.$$

Since column three is not a pivot column, we set

$$\begin{aligned} z &= t \\ y &= \frac{17}{5}z = \frac{17}{5}t \\ x &= 2z - \frac{3}{2}y = 2t - \frac{51}{10}t = -\frac{31}{10}t. \end{aligned}$$

Thus the solution is $\mathbf{x}_h = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{31}{10}t \\ \frac{17}{5}t \\ t \end{bmatrix} = t \begin{bmatrix} -31 \\ 34 \\ 10 \end{bmatrix}.$

[2] (b) The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 7 \\ 4 & 1 & 9 & 5 \\ 0 & 5 & -17 & 9 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -2 & \frac{7}{2} \\ 4 & 1 & 9 & 5 \\ 0 & 5 & -17 & 9 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -2 & \frac{7}{2} \\ 0 & -5 & 17 & -9 \\ 0 & 5 & -17 & 9 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -2 & \frac{7}{2} \\ 0 & 1 & -\frac{17}{5} & \frac{9}{5} \\ 0 & 5 & -17 & 9 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -2 & \frac{7}{2} \\ 0 & 1 & -\frac{17}{5} & \frac{9}{5} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since column three is not a pivot column, we set

$$\begin{aligned} z &= t \\ y &= \frac{9}{5} + \frac{17}{5}z = \frac{9}{5} + \frac{17}{5}t \\ x &= \frac{7}{2} + 2z - \frac{3}{2}y = \frac{7}{2} + 2t - \frac{27}{10} - \frac{51}{10}t = \frac{4}{5} - \frac{31}{10}t. \end{aligned}$$

Thus the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{5} - \frac{31}{10}t \\ \frac{9}{5} + \frac{17}{5}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{9}{5} \\ 0 \end{bmatrix} + t \begin{bmatrix} -31 \\ 34 \\ 10 \end{bmatrix} = \mathbf{x}_p + \mathbf{x}_h.$$

[6] 2. (a) We construct the matrix

$$\begin{bmatrix} 1 & 5 & 0 & 2 \\ 1 & 1 & -4 & -2 \\ 4 & 5 & -3 & -1 \\ -2 & -3 & 3 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - (-2)R_1]{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1}} \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & -4 & -4 & -4 \\ 0 & -15 & -3 & -9 \\ 0 & 7 & 3 & 5 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{4}R_2} \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -15 & -3 & -9 \\ 0 & 7 & 3 & 5 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 7R_2]{R_3 \to R_3 - (-15)R_2} \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 12 & 6 \\ 0 & 0 & -4 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{1}{12}R_3} \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & -4 & -2 \end{bmatrix} \xrightarrow{R_4 \to R_4 - (-4)R_3} \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence there is a non-pivot column, and so these four vectors are linearly <u>dependent</u>.

[2] (b) Because the columns are linearly dependent, the matrix A is <u>not</u> invertible.

[6] 3. (a) We have

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_4 \to R_4 - 3R_1]{R_2 \to R_2 - (-2)R_1} \left[\begin{array}{cccc|cccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -3 & -3 & -2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cccc|cccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3 & -3 & -2 & -3 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to -\frac{1}{3}R_2} \left[\begin{array}{cccc|cccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{3} & 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{3} & -1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 1 & \frac{2}{3} & 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to -R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{3} & -1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 1 & \frac{2}{3} & 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{3} & -1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{8}{3} & 1 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & -2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 \to \frac{1}{2}R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{3} & -1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{8}{3} & 1 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & -2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \to R_2 - \frac{8}{3}R_4 \\ R_3 \to R_3 - (-2)R_4}]{R_1 \to R_1 - \left(-\frac{1}{3}\right)R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{2}{3} & \frac{1}{6} & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 & -\frac{5}{3} & -\frac{4}{3} & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right].$$

Thus

$$A^{-1} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{6} & 0 & \frac{2}{3} \\ -\frac{5}{3} & -\frac{4}{3} & 1 & -\frac{1}{3} \\ 2 & 1 & -1 & 0 \\ 1 & \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

[3] (b) We have

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 2 & 0 & 1 & 0 & 0 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_4 \to R_4 - 3R_1]{R_2 \to R_2 - (-2)R_1} \left[\begin{array}{cccc|cccc} 1 & 2 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -3 & -3 & 1 & -3 & 0 & 0 & 1 \end{array}\right].$$

Because we have a row of zeroes, it is impossible to reduce B to I. Hence B is not invertible.

[7] 4. We have

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & -4 & -5 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - (-2)R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -4 & -7 & 2 & 1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \\ 0 & -4 & -7 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - (-4)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \\ 0 & 0 & 1 & -6 & 1 & 4 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - (-1)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 1 & 4 \\ 0 & 1 & 0 & 10 & -2 & -7 \\ 0 & 0 & 1 & -6 & 1 & 4 \end{array}\right]$$

and so

$$A^{-1} = \begin{bmatrix} -5 & 1 & 4 \\ 10 & -2 & -7 \\ -6 & 1 & 4 \end{bmatrix}.$$

Hence the solution of the system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} -2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 4 \\ 10 & -2 & -7 \\ -6 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 3 \end{bmatrix}.$$

[8] 5. First we bring A to reduced row-echlon form:

$$\begin{bmatrix} 2 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & 1 & 0 \\ 3 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - (-4)R_2]{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The elementary matrices required to do this are

$$E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

The associated inverses are

$$E_1^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$E_4^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

and so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}.$$