MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 5	**MATH 2050**	Winter 2018

SOLUTIONS

[5] 1. (a) The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 10 \\ 2 & -1 & -5 & -5 \\ -3 & 2 & 6 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - (-3)R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 10 \\ 0 & -1 & -13 & -25 \\ 0 & 2 & 18 & 30 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 10 \\ 0 & 1 & 13 & 25 \\ 0 & 2 & 18 & 30 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 10 \\ 0 & 1 & 13 & 25 \\ 0 & 0 & -8 & -20 \end{array}\right]$$

$$\xrightarrow{R_3 \to -\frac{1}{8}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 10 \\ 0 & 1 & 13 & 25 \\ 0 & 0 & 1 & \frac{5}{2} \end{array}\right]$$

so then we see that

$$\begin{aligned} z &= \frac{5}{2} \\ y &= 25 - 13z = 25 - \frac{65}{2} = -\frac{15}{2} \\ x &= 10 - 4z = 10 - 10 = 0. \end{aligned}$$

Thus the unique solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{15}{2} \\ \frac{5}{2} \end{bmatrix}$.

[5] (b) The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 4 & 10 \\ 2 & -1 & -5 & -5 \\ -3 & 2 & 6 & 0 \end{array}\right] \xrightarrow{R_1 \to (-1)R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -10 \\ 2 & -1 & -5 & -5 \\ -3 & 2 & 6 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - (-3)R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -10 \\ 0 & -1 & 3 & 15 \\ 0 & 2 & -6 & -30 \end{array}\right] \xrightarrow{R_2 \to (-1)R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -10 \\ 0 & 1 & -3 & -15 \\ 0 & 2 & -6 & -30 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -10 \\ 0 & 1 & -3 & -15 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the third column is not a pivot column, and we can set

$$\begin{aligned} z &= t \\ y &= -15 + 3z = -15 + 3t \\ x &= -10 + 4z = -10 + 4t \end{aligned}$$

and so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10+4t \\ -15+3t \\ t \end{bmatrix} = \begin{bmatrix} -10 \\ -15 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}.$$

This system has an infinite number of solutions.

[5] (c) The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 4 & -10 \\ 2 & -1 & -5 & -5 \\ -3 & 2 & 6 & 0 \end{array}\right] \xrightarrow{R_1 \to (-1)R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 10 \\ 2 & -1 & -5 & -5 \\ -3 & 2 & 6 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - (-3)R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 10 \\ 0 & -1 & 3 & -25 \\ 0 & 2 & -6 & 30 \end{array}\right] \xrightarrow{R_2 \to (-1)R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 10 \\ 0 & 1 & -3 & 25 \\ 0 & 2 & -6 & 30 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 10 \\ 0 & 1 & -3 & 25 \\ 0 & 0 & 0 & -20 \end{array}\right]$$

and since the last row now implies that $0 = -20$, the system must be inconsistent. Hence there is no solution.

[6] (d) The augmented matrix is

$$\left[\begin{array}{cccc|c} 3 & 12 & -6 & 0 & -15 \\ 2 & 8 & -1 & 3 & -4 \\ -1 & -4 & 6 & 4 & 13 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{3}R_1} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 0 & -5 \\ 2 & 8 & -1 & 3 & -4 \\ -1 & -4 & 6 & 4 & 13 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - (-1)R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 0 & -5 \\ 0 & 0 & 3 & 3 & 6 \\ 0 & 0 & 4 & 4 & 8 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 0 & -5 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & 4 & 8 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 4R_2} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 0 & -5 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The second and fourth columns are non-pivot columns, so we let $z = t$ and $x = s$. Then

$$y = 2 - z = 2 - t$$
$$w = -5 - 4x + 2y = -5 - 4s + (4 - 2t) = -1 - 4s - 2t$$

and so

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1-4s-2t \\ s \\ 2-t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

This system has an infinite number of solutions.

[7] (e) The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 4 \\ 4 & -1 & 5 & 0 & 2 \\ -7 & -3 & 5 & 4 & 7 \\ 2 & 0 & 6 & 1 & -5 \end{array}\right] \xrightarrow[\substack{R_2\to R_2-4R_1 \\ R_3\to R_3-(-7)R_1 \\ R_4\to R_4-2R_1}]{} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 4 \\ 0 & -9 & 1 & -8 & -14 \\ 0 & 11 & 12 & 18 & 35 \\ 0 & -4 & 4 & -3 & -13 \end{array}\right]$$

$$\xrightarrow{R_2\to -\frac{1}{9}R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 4 \\ 0 & 1 & -\frac{1}{9} & \frac{8}{9} & \frac{14}{9} \\ 0 & 11 & 12 & 18 & 35 \\ 0 & -4 & 4 & -3 & -13 \end{array}\right] \xrightarrow[\substack{R_3\to R_3-11R_2 \\ R_4\to R_4-(-4)R_2}]{} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 4 \\ 0 & 1 & -\frac{1}{9} & \frac{8}{9} & \frac{14}{9} \\ 0 & 0 & \frac{119}{9} & \frac{74}{9} & \frac{161}{9} \\ 0 & 0 & \frac{32}{9} & \frac{5}{9} & -\frac{61}{9} \end{array}\right]$$

$$\xrightarrow{R_3\to \frac{9}{119}R_3} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 4 \\ 0 & 1 & -\frac{1}{9} & \frac{8}{9} & \frac{14}{9} \\ 0 & 0 & 1 & \frac{74}{119} & \frac{23}{17} \\ 0 & 0 & \frac{32}{9} & \frac{5}{9} & -\frac{61}{9} \end{array}\right] \xrightarrow{R_4\to R_4-\frac{32}{9}R_3} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 4 \\ 0 & 1 & -\frac{1}{9} & \frac{8}{9} & \frac{14}{9} \\ 0 & 0 & 1 & \frac{74}{119} & \frac{23}{17} \\ 0 & 0 & 0 & -\frac{197}{119} & -\frac{197}{17} \end{array}\right]$$

$$\xrightarrow{R_4\to -\frac{119}{197}R_4} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 4 \\ 0 & 1 & -\frac{1}{9} & \frac{8}{9} & \frac{14}{9} \\ 0 & 0 & 1 & \frac{74}{119} & \frac{23}{17} \\ 0 & 0 & 0 & 1 & 7 \end{array}\right].$$

Thus

$$\begin{aligned} z &= 7 \\ y &= \frac{23}{17} - \frac{74}{119}z = \frac{23}{17} - \frac{74}{17} = -3 \\ x &= \frac{14}{9} - \frac{8}{9}z + \frac{1}{9}y = \frac{14}{9} - \frac{56}{9} - \frac{1}{3} = -5 \\ w &= 4 - 2z - y - 2x = 4 - 14 + 3 + 10 = 3 \end{aligned}$$

and so the unique solution is

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ -3 \\ 7 \end{bmatrix}.$$

[7] 2. We need to determine if there is a solution to the equation $A\mathbf{x} = \mathbf{b}$, so we row-reduce the

augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & -3 & 1 \\ 0 & 3 & 5 & -1 & -4 \\ 1 & -1 & 0 & -6 & 0 \\ 2 & 1 & 7 & -5 & 4 \end{array}\right] \xrightarrow[R_4 \to R_4 - 2R_1]{R_3 \to R_3 - R_1} \left[\begin{array}{cccc|c} 1 & -2 & -1 & -3 & 1 \\ 0 & 3 & 5 & -1 & -4 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 5 & 9 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & -2 & -1 & -3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 3 & 5 & -1 & -4 \\ 0 & 5 & 9 & 1 & 2 \end{array}\right] \xrightarrow[R_4 \to R_4 - 5R_2]{R_3 \to R_3 - 3R_2} \left[\begin{array}{cccc|c} 1 & -2 & -1 & -3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 2 & 8 & -1 \\ 0 & 0 & 4 & 16 & 7 \end{array}\right]$$

$$\xrightarrow{R_3 \to \frac{1}{2}R_3} \left[\begin{array}{cccc|c} 1 & -2 & -1 & -3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 1 & 4 & -\frac{1}{2} \\ 0 & 0 & 4 & 16 & 7 \end{array}\right] \xrightarrow{R_4 \to R_4 - 4R_3} \left[\begin{array}{cccc|c} 1 & -2 & -1 & -3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 1 & 4 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 9 \end{array}\right].$$

Since the last row now implies that $0 = 9$, this system of equations must be inconsistent, and has no solutions. Thus **b** cannot be written as a linear combination of the columns of A.

[5] 3. First we row-reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 1 & 2 & b & 2 \\ 2 & 3 & 0 & c \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & 1 & b-a & 1 \\ 0 & 1 & -2a & c-2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & 1 & b-a & 1 \\ 0 & 0 & -a-b & c-3 \end{array}\right].$$

Note that we've chosen not to write the final pivot as a 1, because we cannot be certain that $-a - b \neq 0$.

(a) To obtain a unique solution, we must ensure that all of the columns are pivot columns, so we need

$$-a - b \neq 0 \quad \text{or} \quad a \neq -b.$$

(b) To obtain an infinite number of solutions, we must have at least one non-pivot column. This can only happen if

$$-a - b = 0 \quad \text{or} \quad a = -b.$$

For consistency, we also need

$$c - 3 = 0 \quad \text{or} \quad c = 3.$$

(c) To obtain an inconsistent system, we must have a row of zeroes on the lefthand side with a non-zero entry on the righthand side. Thus we must have

$$-a - b = 0 \quad \text{or} \quad a = -b$$

and

$$c - 3 \neq 0 \quad \text{or} \quad c \neq 3.$$