MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 4 **MATH 2050** WINTER 2018

SOLUTIONS

[10] 1. (a) First,

$$AB = \begin{bmatrix} 3 & 0 & -2 \\ 5 & -5 & 1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -2 & 0 \\ 7 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 11 \\ 37 & 14 \\ 25 & -3 \end{bmatrix}.$$

However, we cannot compute BA because B has 2 columns while A has 3 rows.

Next,

$$B^T A = \begin{bmatrix} 4 & -2 & 7 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -2 \\ 5 & -5 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 11 \\ 9 & 2 & -9 \end{bmatrix}.$$

Next,

$$A^2 = AA = \begin{bmatrix} 3 & 0 & -2 \\ 5 & -5 & 1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 & -2 \\ 5 & -5 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 4 & -12 \\ -10 & 23 & -12 \\ -10 & 4 & 7 \end{bmatrix}.$$

We cannot compute $B^2 = BB$ because the first matrix in the product has 2 columns, while the second matrix has 3 rows.

Finally,

$$B^T B = \begin{bmatrix} 4 & -2 & 7 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -2 & 0 \\ 7 & -1 \end{bmatrix} = \begin{bmatrix} 69 & 5 \\ 5 & 10 \end{bmatrix}.$$

[3] (b) We have

$$\frac{1}{4}X - 2A = C^T$$

$$\frac{1}{4}X - 2\begin{bmatrix} 3 & 0 & -2 \\ 5 & -5 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -1 & 4 \\ -6 & 13 & 2 \\ 0 & 4 & -5 \end{bmatrix}^T$$

$$\frac{1}{4}X - \begin{bmatrix} 6 & 0 & -4 \\ 10 & -10 & 2 \\ 0 & -4 & 6 \end{bmatrix} = \begin{bmatrix} -4 & -6 & 0 \\ -1 & 13 & 2 \\ 4 & 2 & -5 \end{bmatrix}$$

$$\frac{1}{4}X = \begin{bmatrix} 2 & -6 & -4 \\ 9 & 3 & 6 \\ 4 & -2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 8 & -24 & -16 \\ 36 & 12 & 24 \\ 16 & -8 & 4 \end{bmatrix}.$$

[4] 2. Since A commutes with $A+B$, we know that

$$A(A+B) = (A+B)A.$$

Then, by the distributive property,

$$\begin{aligned} A^2 + AB &= A^2 + BA \\ AB &= BA. \end{aligned}$$

Hence A commutes with B.

[2] 3. (a) The matrix equation is equivalent to

$$\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 9x_1 + 6x_2 &= 3. \end{aligned}$$

[5] (b) Observe that

$$ad - bc = 18 - (-9) = 27,$$

so A is invertible. Thus we have

$$A^{-1} = \frac{1}{27}\begin{bmatrix} 6 & 1 \\ -9 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{1}{27} \\ -\frac{1}{3} & \frac{1}{9} \end{bmatrix}.$$

Thus

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{27}\begin{bmatrix} 6 & 1 \\ -9 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} -3 \\ 18 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{3} \end{bmatrix}.$$

[3] (c) Since

$$\begin{bmatrix} 3 & -1 \\ 9 & 6 \end{bmatrix}\begin{bmatrix} -\frac{1}{9} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

we can write

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix} = -\frac{1}{9}\begin{bmatrix} 3 \\ 9 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} -1 \\ 6 \end{bmatrix}.$$

[4] (d) Now observe that

$$ad - bc = 18 - 18 = 0,$$

so Z is not invertible. However, the corresponding system of equations is now

$$\begin{aligned} 3x_1 - 2x_2 &= -1 \\ -9x_1 + 6x_2 &= 3. \end{aligned}$$

From the first equation, we can see that $2x_2 = 3x_1 + 1$. Substituting this into the second equation, we have

$$-9x_1 + 3(2x_2) = 3 \quad \Longrightarrow \quad -9x_1 + 3(3x_1 + 1) = 3 \quad \Longrightarrow \quad 3 = 3.$$

Since this is true for any value of x_1, we can (for example) choose $x_1 = 1$ and see that $2x_2 = 4$ so $x_2 = 2$. Hence

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

[5] 4. We have

$$\begin{aligned} AX + 4B &= C \\ AX &= C - 4B \\ X &= A^{-1}(C - 4B), \end{aligned}$$

if A is invertible. In fact for A we can see that

$$ad - bc = 0 - (-3) = 3,$$

so it is invertible and

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 0 & -1 \\ 3 & 4 \end{bmatrix}.$$

Thus

$$\begin{aligned} X &= \frac{1}{3}\begin{bmatrix} 0 & -1 \\ 3 & 4 \end{bmatrix}\left(\begin{bmatrix} 3 & 4 & 3 \\ -9 & 7 & 1 \end{bmatrix} - \begin{bmatrix} -8 & 4 & 28 \\ 0 & 4 & -20 \end{bmatrix}\right) \\ &= \frac{1}{3}\begin{bmatrix} 0 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 11 & 0 & -25 \\ -9 & 3 & 21 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 9 & -3 & -21 \\ -3 & 12 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 & -7 \\ -1 & 4 & 3 \end{bmatrix}. \end{aligned}$$

[4] 5. Since B is invertible, we can left-multiply both sides of the equation by B^{-1} to get

$$\begin{aligned} B^{-1}BA^{-1}X^TB &= B^{-1}BA^T \\ IA^{-1}X^TB &= IA^T \\ A^{-1}X^TB &= A^T. \end{aligned}$$

Likewise, we can left-multiply both sides of the equation by A to get

$$\begin{aligned} AA^{-1}X^TB &= AA^T \\ IX^TB &= AA^T \\ X^TB &= AA^T. \end{aligned}$$

We can right-multiply both sides of the equation by B^{-1} to get

$$\begin{aligned} X^TBB^{-1} &= AA^TB^{-1} \\ X^TI &= AA^TB^{-1} \\ X^T &= AA^TB^{-1}. \end{aligned}$$

Finally, using the properties of the matrix transpose,

$$
\begin{aligned}
X &= (AA^T B^{-1})^T \\
&= (B^{-1})^T (A^T)^T A^T \\
&= (B^{-1})^T AA^T.
\end{aligned}
$$

(Here we could alternatively write $(B^{-1})^T = (B^T)^{-1}$.)