MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 3 | **MATH 2050** | WINTER 2018

SOLUTIONS

[3] 1. (a) We have

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{5}{18}\begin{bmatrix}4\\1\\-1\end{bmatrix}.$$

[3] (b) We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{5}{13}\begin{bmatrix}2\\-3\\0\end{bmatrix}.$$

[5] 2. (a) First we need two vectors which lie in π. Since $x = 2y - 3z$, we can write any vector in π in the form

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2y-3z\\y\\z\end{bmatrix} = y\begin{bmatrix}2\\1\\0\end{bmatrix} + z\begin{bmatrix}-3\\0\\1\end{bmatrix},$$

and so both $\mathbf{u} = \begin{bmatrix}2\\1\\0\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}-3\\0\\1\end{bmatrix}$ must lie in π. Now we will project $\mathbf{u}$ onto $\mathbf{v}$ to get

$$\mathbf{p} = \operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{-6}{10}\begin{bmatrix}-3\\0\\1\end{bmatrix} = \frac{3}{5}\begin{bmatrix}3\\0\\-1\end{bmatrix}.$$

Therefore a vector orthogonal to $\mathbf{p}$, and hence also to $\mathbf{v}$, is

$$\mathbf{u} - \mathbf{p} = \begin{bmatrix}\frac{1}{5}\\1\\\frac{3}{5}\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1\\5\\3\end{bmatrix}.$$

To make life easier, a more convenient orthogonal vector is therefore

$$\mathbf{w} = \begin{bmatrix}1\\5\\3\end{bmatrix}.$$

[3] (b) We will use the orthogonal vectors $\mathbf{v}$ and $\mathbf{w}$ we found in part (a). Then

$$\operatorname{proj}_{\pi} \mathbf{t} = \frac{\mathbf{t}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{v}}\mathbf{v} + \frac{\mathbf{t}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w} = \frac{-2}{10}\begin{bmatrix}-3\\0\\1\end{bmatrix} + \frac{-1}{35}\begin{bmatrix}1\\5\\3\end{bmatrix} = \begin{bmatrix}\frac{4}{7}\\-\frac{1}{7}\\-\frac{2}{7}\end{bmatrix} = \frac{1}{7}\begin{bmatrix}4\\-1\\-2\end{bmatrix}.$$

[6] 3. First we need to identify any point Q in π, such as $Q(0,7,0)$. Then the vector

$$\mathbf{u} = \overrightarrow{PQ} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}.$$

Observe that the normal to the plane is the vector $\mathbf{n} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$. Hence we project $\mathbf{u}$ onto $\mathbf{n}$ to get

$$\mathbf{p} = \text{proj}_{\mathbf{n}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} = \frac{-6}{10}\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \frac{3}{5}\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{3}{5} \\ \frac{9}{5} \end{bmatrix}.$$

Finally, we need to identify the point R for which $\mathbf{p} = \overrightarrow{PR}$. If R is the point (x,y,z) then

$$\begin{bmatrix} 0 \\ -\frac{3}{5} \\ \frac{9}{5} \end{bmatrix} = \begin{bmatrix} x+2 \\ y-1 \\ z+4 \end{bmatrix}$$

and so $x = -2$, $y = \frac{2}{5}$ and $z = -\frac{11}{5}$. In other words, the point in π closest to P is $(-2, \frac{2}{5}, -\frac{11}{5})$.

[6] 4. First we need to identify a point Q on ℓ, such as $Q(-1,3,-4)$. Next we construct the vector from Q to the origin O, $\mathbf{u} = \overrightarrow{QO} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$. Observe that the direction vector of ℓ is $\mathbf{d} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Then the projection of $\mathbf{u}$ onto ℓ is

$$\mathbf{p} = \text{proj}_{\mathbf{d}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{d}}{\mathbf{d}\cdot\mathbf{d}}\mathbf{d} = \frac{9}{6}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

and so

$$\mathbf{u} - \mathbf{p} = \begin{bmatrix} -2 \\ -\frac{3}{2} \\ \frac{5}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -4 \\ -3 \\ 5 \end{bmatrix}.$$

Finally, the distance from the origin to ℓ is given by

$$\|\mathbf{u} - \mathbf{p}\| = \frac{1}{2}\sqrt{(-4)^2 + (-3)^2 + 5^2} = \frac{\sqrt{50}}{2} = \frac{5\sqrt{2}}{2}.$$

[4] 5. We set

$$\begin{aligned} \mathbf{u}\cdot(\mathbf{u} + k\mathbf{v}) &= 0 \\ \mathbf{u}\cdot\mathbf{u} + k(\mathbf{u}\cdot\mathbf{v}) &= 0 \\ 78 + 4k &= 0 \\ k &= -\frac{39}{2}. \end{aligned}$$

[5] 6. (a) We set

$$k_1 \begin{bmatrix} 5 \\ -3 \\ 9 \end{bmatrix} + k_2 \begin{bmatrix} -6 \\ -2 \\ -1 \end{bmatrix} + k_3 \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus we have the system of equations

$$\begin{aligned} 5k_1 - 6k_2 - 3k_3 &= 0 \\ -3k_1 - 2k_2 + k_3 &= 0 \\ 9k_1 - k_2 - 4k_3 &= 0. \end{aligned}$$

One way to solve this system is to add 3 times the second equation to the first equation, so

$$-4k_1 - 12k_2 = 0 \quad \Longrightarrow \quad k_1 = -3k_2.$$

Similarly, we could add 3 times the second equation to the third equation, so

$$-7k_2 - k_3 = 0 \quad \Longrightarrow \quad k_3 = -7k_2.$$

Substituting both of these back into the second equation, we have

$$-3(-3k_2) - 2k_2 + (-7k_2) = 0 \quad \Longrightarrow \quad 0 = 0,$$

which must always be true. Hence any value of k_2 satisfies the equation (such as $k_2 = 1$ for which $k_1 = -3$ and $k_3 = -7$). Thus these vectors are linearly dependent.

[5] (b) We set

$$k_1 \begin{bmatrix} 5 \\ 0 \\ -2 \\ 8 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} -1 \\ -4 \\ 2 \\ 0 \end{bmatrix} + k_4 \begin{bmatrix} 2 \\ -4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This results in the system of equations

$$\begin{aligned} 5k_1 - k_3 + 2k_4 &= 0 \\ 3k_2 - 4k_3 - 4k_4 &= 0 \\ -2k_1 + 2k_2 + 2k_3 + k_4 &= 0 \\ 8k_1 + 3k_4 &= 0. \end{aligned}$$

From the fourth equation, we can see that $k_4 = -\frac{8}{3}k_1$. Substituting this into the first equation gives

$$5k_1 - k_3 + 2\left(-\frac{8}{3}k_1\right) = 0 \quad \Longrightarrow \quad -\frac{1}{3}k_1 - k_3 = 0 \quad \Longrightarrow \quad k_3 = -\frac{1}{3}k_1.$$

Substituting both of these into the second equation gives

$$3k_2 - 4\left(-\frac{1}{3}k_1\right) - 4\left(-\frac{8}{3}k_1\right) = 0 \quad \Longrightarrow \quad 3k_2 + 12k_1 = 0 \quad \Longrightarrow \quad k_2 = -4k_1.$$

Finally, substituting all of these into the third equation gives

$$-2k_1 + 2(-4k_1) + 2\left(-\frac{1}{3}k_1\right) + \left(-\frac{8}{3}k_1\right) = 0 \quad \Longrightarrow \quad -\frac{40}{3}k_1 = 0 \quad \Longrightarrow \quad k_1 = 0,$$

and thus $k_2 = k_3 = k_4 = 0$ as well. Hence these vectors are linearly <u>independent</u>.