MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 3.4 | **Math 2050 Worksheet** | WINTER 2018

SOLUTIONS

1. (a) First we find the eigenvalues and eigenvectors of A. We have

$$\det(A-\lambda I)=\begin{vmatrix}5-\lambda & -2\\ 1 & 2-\lambda\end{vmatrix}=\lambda^2-7\lambda+12=(\lambda-4)(\lambda-3)=0,$$

so $\lambda_1=4$ and $\lambda_2=3$. We have two distinct eigenvalues (and hence two linearly independent eigenvectors) for this 2×2 matrix, so A is diagonalizable.
For λ_1, $A-\lambda I$ is the matrix

$$\begin{bmatrix}1 & -2\\ 1 & -2\end{bmatrix}\longrightarrow\begin{bmatrix}1 & -2\\ 0 & 0\end{bmatrix}$$

so if $\mathbf{x}=\begin{bmatrix}x\\ y\end{bmatrix}$ then $y=t$ and $x=2t$. Thus an eigenvector corresponding to λ_1 is $\mathbf{x}_1=\begin{bmatrix}2\\ 1\end{bmatrix}$.
For λ_2, $A-\lambda I$ is the matrix

$$\begin{bmatrix}2 & -2\\ 1 & -1\end{bmatrix}\longrightarrow\begin{bmatrix}2 & -2\\ 0 & 0\end{bmatrix}$$

so $y=t$ and $x=t$. Thus an eigenvector corresponding to λ_2 is $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$.
Hence we can let

$$P=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\quad\text{and}\quad D=\begin{bmatrix}4 & 0\\ 0 & 3\end{bmatrix}.$$

(b) We have

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & 1\\ -1 & -\lambda\end{vmatrix}=\lambda^2-2\lambda+1=(\lambda-1)^2=0$$

so $\lambda=1$ is the only eigenvalue. The matrix $A-\lambda I$ is

$$\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\longrightarrow\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$$

so $y=t$ and $x=t$. Hence the only corresponding eigenvector is $\mathbf{x}=-11$ (and its multiples). So we do not have the requisite two linearly independent eigenvectors, and therefore A is not diagonalizable.

(c) We have

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & -16 & -2 \\ 0 & 5-\lambda & 0 \\ 2 & -8 & -3-\lambda \end{vmatrix}$$
$$= (5-\lambda)[(2-\lambda)(-3-\lambda)+4] = -\lambda^3+4\lambda^2+7\lambda-10$$
$$= (\lambda-1)((\lambda+2)(\lambda-5) = 0,$$

so $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 5$. We have three distinct eigenvalues for this 3×3 matrix, so A must be diagonalizable.

For λ_1, $A - \lambda I$ is

$$\begin{bmatrix} 1 & -16 & -2 \\ 0 & 4 & 0 \\ 2 & -8 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -16 & -2 \\ 0 & 4 & 0 \\ 0 & 24 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -16 & -2 \\ 0 & 1 & 0 \\ 0 & 24 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -16 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so if $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then $z = t$, $y = 0$ and $x = 2t$. Hence $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

For λ_2, $A - \lambda I$ is

$$\begin{bmatrix} 4 & -16 & -2 \\ 0 & 7 & 0 \\ 2 & -8 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -\frac{1}{2} \\ 0 & 7 & 0 \\ 2 & -8 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -\frac{1}{2} \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $z = t$, $y = 0$ and $x = \frac{1}{2}t$. Hence $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

For λ_3, $A - \lambda I$ is

$$\begin{bmatrix} -3 & -16 & -2 \\ 0 & 0 & 0 \\ 2 & -8 & -8 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -8 & -8 \\ 0 & 0 & 0 \\ -3 & -16 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -4 \\ 0 & 0 & 0 \\ -3 & -16 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -4 \\ 0 & 0 & 0 \\ 0 & -28 & -14 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & -4 & -4 \\ 0 & -28 & -14 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

so $z = t$, $y = -\frac{1}{2}t$ and $x = 2t$. Hence $\mathbf{x}_3 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$.

Hence we can let

$$P = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & -1 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

(d) We have

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 8-\lambda & 9 & -9 \\ 0 & 2-\lambda & 0 \\ 4 & 6 & -4-\lambda \end{vmatrix} \\ &= (2-\lambda)[(8-\lambda)(-4-\lambda)+36] = -\lambda^3+6\lambda^2-12\lambda+8 \\ &= -(\lambda-2)^3 = 0\end{aligned}$$

so $\lambda = 2$ is the only eigenvalue. The matrix $A - \lambda I$ is

$$\begin{bmatrix} 6 & 9 & -9 \\ 0 & 0 & 0 \\ 4 & 6 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 4 & 6 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $z = t$, $y = s$ and $x = \frac{3}{2}t - \frac{3}{2}s$. Thus we have two linearly independent eigenvectors:

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}.$$

But we require three such eigenvectors, so A is not diagonalizable.