MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTIONS 2.5 & 2.6 **Math 2050 Worksheet** WINTER 2018

SOLUTIONS

1. (a) We have

$$\left[\begin{array}{ccc|ccc} 4 & -8 & 0 & 1 & 0 & 0 \\ 12 & -23 & 0 & 0 & 1 & 0 \\ 0 & 20 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{4}R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & \frac{1}{4} & 0 & 0 \\ 12 & -23 & 0 & 0 & 1 & 0 \\ 0 & 20 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 12R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 20 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 20R_2]{R_1 \to R_1 - (-2)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{23}{4} & 2 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 4 & 60 & -20 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to \frac{1}{4}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{23}{4} & 2 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 15 & -5 & \frac{1}{4} \end{array}\right]$$

so

$$A^{-1} = \begin{bmatrix} -\frac{23}{4} & 2 & 0 \\ -3 & 1 & 0 \\ 15 & -5 & \frac{1}{4} \end{bmatrix}.$$

(b) We have

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -6 & -3 & 1 & 0 \\ 0 & -1 & 6 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - (-1)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -6 & -3 & 1 & 0 \\ 0 & 0 & 0 & -4 & 1 & 1 \end{array}\right].$$

Now we have a row of zeros, so there is no way to row-reduce the matrix on the left to I. Hence A is not invertible.

(c) We have

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & -2 & 1 & 0 & 0 & 0 \\ -4 & 1 & -8 & 8 & 0 & 1 & 0 & 0 \\ 6 & 0 & 19 & -12 & 0 & 0 & 1 & 0 \\ 0 & -2 & -8 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 - (-4)R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -6 & 0 & 1 & 0 \\ 0 & -2 & -8 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - (-2)R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 8 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 4R_3]{R_1 \to R_1 - 3R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & 19 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 28 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 & -6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 8 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - (-2)R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 35 & 4 & -3 & 2 \\ 0 & 1 & 0 & 0 & 28 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 & -6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 8 & 2 & 0 & 1 \end{array}\right]$$

so then

$$A^{-1} = \begin{bmatrix} 35 & 4 & -3 & 2 \\ 28 & 1 & -4 & 0 \\ -6 & 0 & 1 & 0 \\ 8 & 2 & 0 & 1 \end{bmatrix}.$$

2. (a) Writing the system in the form $A\mathbf{x} = \mathbf{b}$ we have

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 5 & \frac{1}{3} & -15 \\ -1 & 1 & 4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}.$$

First we compute A^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 5 & \frac{1}{3} & -15 & 0 & 1 & 0 \\ -1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 \to (-1)R_1]{R_2 \to R_2 - 5R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & -5 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -15 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -15 & 3 & 0 \\ 0 & 0 & 1 & 16 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - (-3)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 49 & -9 & 3 \\ 0 & 1 & 0 & -15 & 3 & 0 \\ 0 & 0 & 1 & 16 & -3 & 1 \end{array}\right]$$

so

$$A^{-1} = \begin{bmatrix} 49 & -9 & 3 \\ -15 & 3 & 0 \\ 16 & -3 & 1 \end{bmatrix}$$

and hence

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 49 & -9 & 3 \\ -15 & 3 & 0 \\ 16 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 1 \end{bmatrix}.$$

(b) The matrix A in this case would be 3×4 and hence not square. Therefore A would not be invertible.

3. First we try to row-reduce A to I:

$$\begin{bmatrix} 4 & -6 \\ 1 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 \\ 4 & -6 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{2}R_2} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - (-1)R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the elementary matrices involved are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The associated inverses are

$$E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad E_4^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

and so

$$E_4E_3E_2E_1A = I \quad \Longrightarrow \quad A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

4. We first row-reduce the matrix of coefficients A to row-echelon form using only the third elementary row operation:

$$A = \begin{bmatrix} 5 & 2 & -1 \\ 1 & 0 & 4 \\ -2 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - \left(-\frac{2}{5}\right) R_1]{R_2 \to R_2 - \frac{1}{5} R_1} \begin{bmatrix} 5 & 2 & -1 \\ 0 & -\frac{2}{5} & \frac{21}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - (-2) R_2} \begin{bmatrix} 5 & 2 & -1 \\ 0 & -\frac{2}{5} & \frac{21}{5} \\ 0 & 0 & 9 \end{bmatrix} = U.$$

Now we compose the matrix L. We subtracted $\frac{1}{5}$ times the 1st row from the 2nd row, so the $(2,1)$ element is $\frac{1}{5}$. We subtracted $-\frac{2}{5}$ times the 1st row from the 3rd row, so the $(3,1)$ element is $-\frac{2}{5}$. We subtracted -2 times the 2nd row from the 3rd row, so the $(3,2)$ element is -2. Hence

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ -\frac{2}{5} & -2 & 1 \end{bmatrix}.$$

So now we want to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 12 \\ -13 \\ -1 \end{bmatrix}$. This is equivalent to $LU\mathbf{x} = \mathbf{b}$, and so first we solve $L\mathbf{y} = \mathbf{b}$ by forward-substitution. We have

$$\begin{aligned} y_1 &= 12 \\ y_2 &= -13 - \frac{1}{5} y_1 = -\frac{77}{5}, \\ y_3 &= -1 + 2y_2 + \frac{2}{5} y_1 = -1 - \frac{154}{5} + \frac{24}{5} = -27. \end{aligned}$$

Now we use back-substitution to solve $U\mathbf{x} = \mathbf{y}$ and so

$$\begin{aligned} x_3 &= \frac{1}{9}(-27) = -3 \\ x_2 &= -\frac{5}{2}\left(-\frac{77}{5} - \frac{21}{5} x_3\right) = 7 \\ x_1 &= \frac{1}{5}(12 + x_3 - 2x_2) = -1. \end{aligned}$$