MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 2.4 | **Math 2050 Worksheet** | WINTER 2018

SOLUTIONS

1. (a) In matrix form, we have

$$\begin{bmatrix} 1 & -2 & -2 \\ -4 & 8 & 6 \end{bmatrix} \xrightarrow{R_2 \to R_2-(-4)R_1} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & -2 \end{bmatrix}$$
$$\xrightarrow{R_2 \to -\frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

so we can let $y = t$ and thus we have

$$z = 0$$
$$x = 2y + 2z = 2t.$$

Hence a solution to the system is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

(b) In matrix form, we have

$$\begin{bmatrix} 1 & -3 & 0 & 4 \\ -1 & 1 & 4 & -2 \\ 1 & 0 & -6 & 1 \\ 2 & -5 & -2 & 7 \end{bmatrix} \xrightarrow[R_4 \to R_4-2R_1]{\substack{R_2 \to R_2-(-1)R_1 \\ R_3 \to R_3-R_1}} \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & -2 & 4 & 2 \\ 0 & 3 & -6 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$
$$\xrightarrow{R_2 \to -\frac{1}{2}R_2} \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix} \xrightarrow{\substack{R_3 \to R_3-3R_2 \\ R_4 \to R_4-R_2}} \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so we can let $x_4 = t$, $x_3 = s$ and then determine

$$x_2 = 2x_3 + x_4 = 2s + t$$
$$x_1 = 3x_2 - 4x_4 = 6s - t.$$

Hence a solution to the system is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6s-t \\ 2s+t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 6 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

2. (a) Using the same matrix manipulations as in #1(a), we have

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -5 \\ -4 & 8 & 6 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & -5 \\ 0 & 0 & -2 & -11 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & -5 \\ 0 & 0 & 1 & \frac{11}{2} \end{array}\right]$$

so we can let $y = t$ and then

$$z = \frac{11}{2}$$
$$x = -5 + 2y + 2z = 6 + 2t$$

so then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6+2t \\ t \\ \frac{11}{2} \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ \frac{11}{2} \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{x}_p + \mathbf{x}_h.$$

(b) Using the same matrix manipulations as in #1(b), we have

$$\left[\begin{array}{cccc|c} 1 & -3 & - & 4 & 6 \\ -1 & 1 & 4 & -2 & -8 \\ 1 & 0 & -6 & 1 & 9 \\ 2 & -5 & -2 & 7 & 13 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -3 & 0 & 4 & 6 \\ 0 & -2 & 4 & 2 & -2 \\ 0 & 3 & -6 & -3 & 3 \\ 0 & 1 & -2 & -1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & -3 & 0 & 4 & 6 \\ 0 & 1 & -2 & -1 & 1 \\ 0 & 3 & -6 & -3 & 3 \\ 0 & 1 & -2 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -3 & 0 & 4 & 6 \\ 0 & 1 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so again we can set $x_4 = t$ and $x_3 = s$ and then we get

$$x_2 = 1 + 2x_3 + x_4 = 1 + 2s + t$$
$$x_1 = 6 + 3x_2 - 4x_4 = 9 + 6s - t$$

so that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9+6s-t \\ 1+2s+t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \left(s \begin{bmatrix} 6 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right) = \mathbf{x}_p + \mathbf{x}_h.$$

3. (a) We let $A\mathbf{k} = \mathbf{0}$. The corresponding matrix is

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -4 & -3 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{4}R_2} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & \frac{3}{4} \\ 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - (-1)R_2} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} \xrightarrow{R_3 \to (-4)R_3} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}$$

so we see that $k_3 = 0$ so $k_2 = -\frac{3}{4}k_3 = 0$ and $k_1 = -3k_2 - 5k_3 = 0$. Hence these three vectors are linearly independent.

(b) We let $A\mathbf{k} = \mathbf{0}$. The corresponding matrix is

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 7 & 5 \\ 1 & -5 & -1 \\ 2 & -6 & 0 \\ 2 & 0 & 3 \end{bmatrix} \xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1 \\ R_5 \to R_5 - 2R_1}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 3 \\ 0 & -6 & -3 \\ 0 & -8 & -4 \\ 0 & -2 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{6}R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & -6 & -3 \\ 0 & -8 & -4 \\ 0 & -2 & -1 \end{bmatrix} \xrightarrow{\substack{R_3 \to R_3 - (-6)R_2 \\ R_4 \to R_4 - (-8)R_2 \\ R_5 \to R_5 - (-2)R_2}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so each of k_3, k_4 and k_5 is a free variable, and hence these vectors are linearly dependent.

(c) We let $A\mathbf{k} = \mathbf{0}$. The corresponding matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 4 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & -4 \end{bmatrix} \xrightarrow{\substack{R_3 \to R_3 - (-1)R_1 \\ R_4 \to R_4 - (-1)R_1}} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 4 & 4 & 4 \\ 0 & 2 & 4 & 6 \\ 0 & 2 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 2 & 2 & 2 \end{bmatrix} \xrightarrow{\substack{R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 2R_2}} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so without further work, we see that k_4 is a free variable. Hence there is an infinite number of solutions to the system, and so these vectors are linearly dependent.

4. We set $AB\mathbf{x} = \mathbf{0}$; we want to show that $\mathbf{x} = \mathbf{0}$ is the only solution to this equation. But note that we can write $AB\mathbf{x} = A\mathbf{y}$ where $\mathbf{y} = B\mathbf{x}$. Then since the columns of A are linearly independent and $A\mathbf{y} = \mathbf{0}$, it must be that $\mathbf{y} = \mathbf{0}$. But then $B\mathbf{x} = \mathbf{0}$, and since the columns of B are linearly independent, it must be that $\mathbf{x} = \mathbf{0}$. Hence the columns of AB are also linearly independent.

   Alternatively, we might recall that a matrix has linearly independent columns if and only if it is invertible. Thus A and B are both invertible, and so if

$$\begin{aligned} AB\mathbf{x} &= \mathbf{0} \\ A^{-1}AB\mathbf{x} &= A^{-1}\mathbf{0} \\ B\mathbf{x} &= \mathbf{0} \\ B^{-1}B\mathbf{x} &= B^{-1}\mathbf{0} \\ \mathbf{x} &= \mathbf{0}, \end{aligned}$$

   again showing that the columns of AB are linearly independent.