SOLUTIONS

1. (a) The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & -5 \\ 4 & -2 & -3 & 1 \\ -2 & 1 & 2 & 0 \end{array}\right] \xrightarrow[R_3\to R_3-(-2)R_1]{R_2\to R_2-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 6 & -5 \\ 0 & -6 & -27 & 21 \\ 0 & 3 & 14 & -10 \end{array}\right]$$

$$\xrightarrow{R_2\to -\frac{1}{6}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 6 & -5 \\ 0 & 1 & \frac{9}{2} & -\frac{7}{2} \\ 0 & 3 & 14 & -10 \end{array}\right] \xrightarrow{R_3\to R_3-3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 6 & -5 \\ 0 & 1 & \frac{9}{2} & -\frac{7}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_3\to 2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 6 & -5 \\ 0 & 1 & \frac{9}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

so then we see that

$$\begin{aligned} z &= 1 \\ y &= -\frac{7}{2} - \frac{9}{2}z = -\frac{7}{2} - \frac{9}{2}(1) = -8 \\ x &= -5 - 6z - y = -5 - 6(1) - (-8) = -3. \end{aligned}$$

Thus the solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \\ 1 \end{bmatrix}$.

(b) The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 5 \\ -1 & 3 & -6 & -13 \\ 1 & 1 & 2 & -3 \end{array}\right] \xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-(-1)R_1} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 5 \\ 0 & 2 & -2 & -8 \\ 0 & 2 & -2 & -8 \end{array}\right]$$

$$\xrightarrow{R_2\to \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 2 & -2 & -8 \end{array}\right] \xrightarrow{R_3\to R_3-2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence x_3 is not determined, and we let

$$\begin{aligned} x_3 &= t \\ x_2 &= -4 + x_3 = -4 + t \\ x_1 &= 5 - 4x_3 + x_2 = 5 - 4t + (-4 + t) = 1 - 3t \end{aligned}$$

so the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-3t \\ -4+t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}.$$

This system has an infinite number of solutions.

(c) The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 6 & -4 & 4 \\ -3 & -4 & 1 & -6 \\ -1 & 2 & -3 & 2 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ -3 & -4 & 1 & -6 \\ -1 & 2 & -3 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - (-1)R_1]{R_2 \to R_2 - (-3)R_1} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & 5 & -5 & 0 \\ 0 & 5 & -5 & 4 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 5 & -5 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

so since the final line implies that $0 = 4$, there is no solution to the system and hence it is inconsistent.

(d) The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 8 \\ 3 & 6 & 1 & -2 & 23 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 8 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 8 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So, first, we see that x_4 is not determined and hence we let $x_4 = t$. Then

$$x_3 = -1 - x_4 = -1 - t.$$

Similarly, x_2 is not determined so we let $x_2 = s$. Finally,

$$x_1 = 8 + x_4 - 2x_2 = 8 + t - 2s.$$

So the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 + t - 2s \\ s \\ -1 - t \\ t \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

This system has an infinite number of solutions.

(e) The augmented matrix is

$$\left[\begin{array}{cccc|c} 3 & 1 & -5 & 2 & 2 \\ 0 & -2 & 1 & -1 & -10 \\ 1 & 4 & -4 & 8 & 8 \\ -3 & -1 & 7 & 0 & -6 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 4 & -4 & 8 & 8 \\ 0 & -2 & 1 & -1 & -10 \\ 3 & 1 & -5 & 2 & 2 \\ -3 & -1 & 7 & 0 & -6 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - (-3)R_1]{R_3 \to R_3 - 3R_1} \left[\begin{array}{cccc|c} 1 & 4 & -4 & 8 & 8 \\ 0 & -2 & 1 & -1 & -10 \\ 0 & -11 & 7 & -22 & -22 \\ 0 & 11 & -5 & 24 & 18 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{2} R_2} \left[\begin{array}{cccc|c} 1 & 4 & -4 & 8 & 8 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 5 \\ 0 & -11 & 7 & -22 & -22 \\ 0 & 11 & -5 & 24 & 18 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - 11R_2]{R_3 \to R_3 - (-11)R_2} \left[\begin{array}{cccc|c} 1 & 4 & -4 & 8 & 8 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 5 \\ 0 & 0 & \frac{3}{2} & -\frac{33}{2} & 33 \\ 0 & 0 & \frac{1}{2} & \frac{37}{2} & -37 \end{array}\right] \xrightarrow{R_3 \to \frac{2}{3} R_3} \left[\begin{array}{cccc|c} 1 & 4 & -4 & 8 & 8 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 5 \\ 0 & 0 & 1 & -11 & 22 \\ 0 & 0 & \frac{1}{2} & \frac{37}{2} & -37 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - \frac{1}{2} R_3} \left[\begin{array}{cccc|c} 1 & 4 & -4 & 8 & 8 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 5 \\ 0 & 0 & 1 & -11 & 22 \\ 0 & 0 & 0 & 24 & -48 \end{array}\right] \xrightarrow{R_4 \to \frac{1}{24} R_4} \left[\begin{array}{cccc|c} 1 & 4 & -4 & 8 & 8 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 5 \\ 0 & 0 & 1 & -11 & 22 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

so we see that

$$\begin{aligned} x_4 &= -2 \\ x_3 &= 22 + 11x_4 = 22 + 11(-2) = 0 \\ x_2 &= 5 - \frac{1}{2}x_4 + \frac{1}{2}x_3 = 5 - \frac{1}{2}(-2) + 0 = 6 \\ x_1 &= 8 - 8x_4 + 4x_3 - 4x_2 = 8 - 8(-2) + 0 - 4(6) = 0 \end{aligned}$$

and hence the unique solution to the system is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ -2 \end{bmatrix}$.

2. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -3 & -5 & p \\ 4 & -10 & -16 & 0 \\ 0 & -4 & -8 & q \end{array}\right] \xrightarrow{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & -3 & -5 & p \\ 0 & 2 & 4 & -4p \\ 0 & -4 & -8 & q \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2} R_2} \left[\begin{array}{ccc|c} 1 & -3 & -5 & p \\ 0 & 1 & 2 & -2p \\ 0 & -4 & -8 & q \end{array}\right] \xrightarrow{R_3 \to R_3 - (-4)R_2} \left[\begin{array}{ccc|c} 1 & -3 & -5 & p \\ 0 & 1 & 2 & -2p \\ 0 & 0 & 0 & q - 8p \end{array}\right].$$

Now we see that the system will be inconsistent if the final row implies $0 = q - 8p$ with $q - 8p \neq 0$. In other words, the desired condition is $q \neq 8p$.

3. A polynomial of degree 2 has the general form $p(x) = ax^2 + bx + c$. We know that $p(1) = a+b+c = -4$, $p(-1) = a-b+c = -12$, and $p(4) = 16a+4b+c = -7$. Writing these three equations as a system (with unknowns a, b and c) yields the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 1 & -1 & 1 & -12 \\ 16 & 4 & 1 & -7 \end{array}\right] \xrightarrow[R_3-16R_1]{R_2\to R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 0 & -2 & 0 & -8 \\ 0 & -12 & -15 & 57 \end{array}\right]$$

$$\xrightarrow{R_2\to -\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & -12 & -15 & 57 \end{array}\right] \xrightarrow{R_3-(-12)R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -15 & 105 \end{array}\right]$$

$$\xrightarrow{R_3\to -\frac{1}{15}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

and so we have that

$$\begin{aligned} c &= -7 \\ b &= 4 \\ a &= -4 - c - b = -4 - (-7) - 4 = -1. \end{aligned}$$

Hence the polynomial is $p(x) = -x^2 + 4x - 7$.