MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 2.1 **Math 2050 Worksheet** WINTER 2013

SOLUTIONS

1. (a) Matrix A has 2 columns ($\mathbf{u}$ and $\mathbf{v}$), each of which has 4 components, meaning A has 4 rows. Hence A is a 4×2 matrix. Matrix B has 2 rows ($\mathbf{u}^T$ and $\mathbf{v}^T$), each of which has 4 components, meaning B has 4 columns. Hence B is a 2×4 matrix.

   (b) Explicitly, we have

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 0 \\ -1 & 2 \\ 7 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -1 & -1 & 7 \\ 2 & 0 & 2 & 0 \end{bmatrix}.$$

   So:

   - $a_{11} = 4$ (the element in the first row, first column);
   - a_{33} does not exist (A has three rows, but does not have three columns);
   - $a_{42} = 0$ (the element in the fourth row, second column);
   - $b_{12} = -1$ (the element in the first row, second column);
   - $b_{21} = 2$ (the element in the second row, first column);
   - b_{42} does not exist (B has two columns, but does not have four rows).

2. Let $A = \begin{bmatrix} 4 & -3 & -1 & 1 \\ 0 & 6 & 0 & 2 \\ -1 & 5 & -1 & -\frac{7}{3} \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}$ so then we can write the system as the matrix equation

$$\begin{bmatrix} 4 & -3 & -1 & 1 \\ 0 & 6 & 0 & 2 \\ -1 & 5 & -1 & -\frac{7}{3} \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}.$$

3. Substituting A and B into the equation yields

$$\begin{bmatrix} 2 & 0 & -4 \\ -1 & -1 & 7 \\ 0 & 8 & 6 \end{bmatrix} - 4X = \frac{1}{3} \begin{bmatrix} 2 & 1 & 3 \\ -2 & 6 & 0 \\ 0 & -3 & 9 \end{bmatrix}^T = \frac{1}{3} \begin{bmatrix} 2 & -2 & 0 \\ 1 & 6 & -3 \\ 3 & 0 & 9 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & 0 \\ \frac{1}{3} & 2 & -1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$4X = \begin{bmatrix} 2 & 0 & -4 \\ -1 & -1 & 7 \\ 0 & 8 & 6 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & 0 \\ \frac{1}{3} & 2 & -1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} & -4 \\ -\frac{4}{3} & -3 & 8 \\ -1 & 8 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & -1 \\ -\frac{1}{3} & -\frac{3}{4} & 2 \\ -\frac{1}{4} & 2 & \frac{3}{4} \end{bmatrix}.$$

4. (a) AB cannot be computed because A has 2 columns while B has 3 rows.

(b) $BA = \begin{bmatrix} 4 & 0 & -4 \\ 3 & -2 & 1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -5 & 6 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 14 & 2 \\ 9 & -16 \end{bmatrix}$

(c) $A^T B = \begin{bmatrix} 1 & -5 & 1 \\ 4 & 6 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 & -4 \\ 3 & -2 & 1 \\ -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -12 & 8 & -9 \\ 32 & -16 & -10 \end{bmatrix}$

(d) $AC = \begin{bmatrix} 1 & 4 \\ -5 & 6 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 6 & 1 & 0 & -2 \\ 1 & \frac{3}{2} & 0 & -8 \end{bmatrix} = \begin{bmatrix} 10 & 7 & 0 & -34 \\ -24 & 4 & 0 & -38 \\ 8 & 4 & 0 & -18 \end{bmatrix}$

(e) $C^T A^T = \begin{bmatrix} 6 & 1 \\ 1 & \frac{3}{2} \\ 0 & 0 \\ -2 & -8 \end{bmatrix} \begin{bmatrix} 1 & -5 & 1 \\ 4 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -24 & 8 \\ 7 & 4 & 4 \\ 0 & 0 & 0 \\ -34 & -38 & -18 \end{bmatrix}$

(f) $B^2 = \begin{bmatrix} 4 & 0 & -4 \\ 3 & -2 & 1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & -4 \\ 3 & -2 & 1 \\ -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 8 & -16 \\ 5 & 2 & -14 \\ -10 & 4 & 2 \end{bmatrix}$

(g) C^2 does not exist, because C has 4 columns but only 2 rows.

(h) $BAC = (BA)C = \begin{bmatrix} 0 & 8 \\ 14 & 2 \\ 9 & -16 \end{bmatrix} \begin{bmatrix} 6 & 1 & 0 & -2 \\ 1 & \frac{3}{2} & 0 & -8 \end{bmatrix} = \begin{bmatrix} 8 & 12 & 0 & -64 \\ 86 & 17 & 0 & -44 \\ 38 & -15 & 0 & 110 \end{bmatrix}$

(i) ACA does not exist because the matrix AC is a 3×4 matrix, and hence has 4 columns, while A has 3 rows.

5. There are many such matrices. For example, let $A = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. The important thing to note is that this means that if $AB = \mathbf{0}$ it does *not* necessarily mean that $A = \mathbf{0}$ or $B = \mathbf{0}$, unlike for real numbers!

6. We let

$$\begin{bmatrix} -10 \\ 13 \\ -10 \end{bmatrix} = x \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}.$$

Then $4x - y = -10$, $y + 3z = 13$ and $-2z = -10$. The last equation implies that $z = 5$ and thus $y = 13 - 3(5) = -2$, and $4x = -2 - 10 = -12$ so $x = -3$. Hence we can write

$$\begin{bmatrix} -10 \\ 13 \\ -10 \end{bmatrix} = -3 \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}.$$