MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 1 | **MATH 2050** | WINTER 2018

SOLUTIONS

[3] 1. (a) The norm of $\mathbf{u}$ is

$$\|\mathbf{u}\| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7.$$

Hence a unit vector in the direction of $\mathbf{u}$ is

$$\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{7}\begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix},$$

and so a unit vector in the opposite direction to $\mathbf{u}$ must be

$$-\frac{1}{7}\begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}.$$

[3] (b) We know that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta).$$

We have already found that $\|\mathbf{u}\| = 7$, while

$$\|\mathbf{v}\| = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{5} \quad \text{and} \quad \mathbf{u} \cdot \mathbf{v} = 2(1) + -3(0) + 6(2) = 14.$$

There

$$14 - 7\sqrt{5}\cos(\theta) \quad \Longrightarrow \quad \cos(\theta) = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$

[5] (c) We must determine if there exist scalars k and ℓ such that $k\mathbf{u} + \ell\mathbf{v} = \mathbf{w}$, and thus

$$\begin{aligned} 2k + \ell &= 1 \\ -3k &= 6 \\ 6k + 2\ell &= -2. \end{aligned}$$

From the second equation, we immediately have $k = -2$. Substituting this into the first equation gives $-4 + \ell = 1$ so $\ell = 5$. We must ensure that these values also satisfy the third equation, and so we have $6k + 2\ell = 6(-2) + 2(5) = -2$ as required. Thus $k = -2$ and $\ell = 5$ is a solution of the system, which means that $-2\mathbf{u} + 5\mathbf{v} = \mathbf{w}$ and therefore $\mathbf{w}$ lies in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

[3] (d) Such a normal is given by the cross product of the vectors:

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} 0 & 6 \\ 2 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 0 & 6 \end{vmatrix}\mathbf{k} = -12\mathbf{i} - (-4)\mathbf{j} + 6\mathbf{k} = \begin{bmatrix} -12 \\ 4 \\ 6 \end{bmatrix}.$$

[3] 2. (a) Since ℓ is perpendicular to π, the normal to π must be the direction vector of ℓ. Hence

$$\mathbf{d} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

and so the equation of the line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

[5] (b) The point Q must satisfy both the equation of the line and the equation of the plane. The parametric equations of the line indicate that $x = 1 + t$, $y = 3 + t$ and $z = 6 - 2t$. Substituting these into the equation of the plane, we have

$$\begin{aligned} (1+t) + (3+t) - 2(6-2t) &= 4 \\ -8 + 6t &= 4 \\ t &= 2. \end{aligned}$$

Thus the point of intersection is $Q(3, 5, 2)$.

[6] (c) First we need a point in the plane, such as the point Q we found in part (b). Now we'll define $\mathbf{u}$ to be the vector which starts at P and ends at Q so

$$\mathbf{u} = \overrightarrow{PQ} = \begin{bmatrix} 3-(-4) \\ 5-0 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}.$$

Now we will project $\mathbf{u}$ onto the normal $\mathbf{n}$ of the plane to obtain the vector $\mathbf{p}$ as follows:

$$\mathbf{p} = \operatorname{proj}_{\mathbf{n}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} = \frac{7(1) + 5(1) + 0(-2)}{1^2 + 1^2 + (-2)^2} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \frac{12}{6} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}.$$

Thus the distance from P to the plane is given by

$$\|\mathbf{p}\| = \sqrt{2^2 + 2^2 + (-4)^2} = \sqrt{24} = 2\sqrt{6}.$$

3. We set $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = \mathbf{0}$ so

$$\begin{aligned} k_1 - 3k_2 - k_3 &= 0 \\ -5k_2 - k_3 &= 0 \\ -5k_1 + 2k_3 &= 0 \\ -4k_1 + 8k_2 &= 0. \end{aligned}$$

From the second and fourth equations we obtain $k_3 = -5k_2$ and $k_1 = 2k_2$. Substituting these into the third equation yields

$$-5(2k_2) + 2(-5k_2) = 0 \implies -20k_2 = 0 \implies k_2 = 0$$

and therefore $k_1 = k_3 = 0$ as well. (We can check that these values satisfy the first equation, but of course the trivial solution is always a solution to this system.) This means that the only solution to the given system of equations is the trivial solution, and so these vectors are linearly independent.

[6] 4. (a) Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

(b) Vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent if $k\mathbf{u} + \ell\mathbf{v} = \mathbf{0}$ implies that $k = \ell = 0$.

(c) We set

$$k\mathbf{u} + \ell\mathbf{v} = \mathbf{0}$$

and we wish to show that this implies that $k = \ell = 0$. We are given that $\mathbf{u} \cdot \mathbf{v} = 0$ and that $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors. Taking the dot product with $\mathbf{v}$ on both sides of the equation gives

$$\begin{aligned} (k\mathbf{u} + \ell\mathbf{v}) \cdot \mathbf{v} &= \mathbf{0} \cdot \mathbf{v} \\ k\mathbf{u} \cdot \mathbf{v} + \ell\mathbf{v} \cdot \mathbf{v} &= 0 \\ k(0) + \ell\|\mathbf{v}\|^2 &= 0 \\ \ell\|\mathbf{v}\|^2 &= 0. \end{aligned}$$

Since $\mathbf{v}$ is a non-zero vector, $\|\mathbf{v}\| \neq 0$ and therefore it must be that $\ell = 0$. Likewise, if we take the dot product with $\mathbf{u}$ on both sides of the equation, we have

$$\begin{aligned} (k\mathbf{u} + \ell\mathbf{v}) \cdot \mathbf{u} &= \mathbf{0} \cdot \mathbf{v} \\ k\mathbf{u} \cdot \mathbf{u} + \ell\mathbf{v} \cdot \mathbf{u} &= 0 \\ k\|\mathbf{u}\|^2 + \ell(0) &= 0 \\ k\|\mathbf{u}\|^2 &= 0 \end{aligned}$$

and so $k = 0$. Since it must be that $k = \ell = 0$, we have proved that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.