MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 1 | **MATH 2050** | WINTER 2018

SOLUTIONS

[5] 1. The vectors are parallel if

$$\begin{bmatrix} 9 \\ x^2 \\ 2x \end{bmatrix} = k \begin{bmatrix} 6 \\ 24 \\ -8 \end{bmatrix}$$

for some scalar k. Thus we must have $9 = 6k$, $x^2 = 24k$ and $2x = -8k$. From the first equation, we immediately have $k = \frac{9}{6} = \frac{3}{2}$. Then, from the second equation, we must have

$$x^2 = 24\left(\frac{3}{2}\right) \quad \Longrightarrow \quad x^2 = 36 \quad \Longrightarrow \quad x = \pm 6.$$

Finally, from the third equation, we must have

$$2x = -8\left(\frac{3}{2}\right) \quad \Longrightarrow \quad x = -6.$$

Since all three equations must be satisfied, the only such value of x is therefore $x = -6$.

[4] 2. We wish to express

$$\begin{bmatrix} 5 \\ 0 \end{bmatrix} = a \begin{bmatrix} 7 \\ -1 \end{bmatrix} + b \begin{bmatrix} -3 \\ 4 \end{bmatrix},$$

which gives us the system of equations

$$\begin{aligned} 7a - 3b &= 5 \\ -a + 4b &= 0. \end{aligned}$$

One way to solve this system is to solve the second equation for a, giving $a = 4b$. Then we can substitute this into the first equation, yielding

$$7(4b) - 3b = 5 \quad \Longrightarrow \quad 25b = 5 \quad \Longrightarrow \quad b = \frac{1}{5} \quad \Longrightarrow \quad a = \frac{4}{5}.$$

Hence

$$\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{4}{5} \begin{bmatrix} 7 \\ -1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

[5] 3. (a) We wish to express

$$\begin{bmatrix} 4 \\ 0 \\ -6 \end{bmatrix} = a \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} + c \begin{bmatrix} -5 \\ 2 \\ -1 \end{bmatrix}.$$

This results in the system of equations

$$2a - 5c = 4$$
$$4b + 2c = 0$$
$$a + 2b - c = -6.$$

One approach is to use the first and second equations to find

$$a = 2 + \frac{5}{2}c \quad \text{and} \quad b = -\frac{1}{2}c.$$

We can substitute both of these into the third equation, giving

$$\left(2 + \frac{5}{2}c\right) + 2\left(-\frac{1}{2}c\right) - c = -6 \quad \Longrightarrow \quad \frac{1}{2}c = -8 \quad \Longrightarrow \quad c = -16.$$

Thus $a = -38$ and $b = 8$. This means that $\left[\begin{smallmatrix} 4 \\ 0 \\ -6 \end{smallmatrix}\right]$ is a linear combination of the vectors, and can be written

$$\begin{bmatrix} 4 \\ 0 \\ -6 \end{bmatrix} = -38 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 8 \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} - 16 \begin{bmatrix} -5 \\ 2 \\ -1 \end{bmatrix}.$$

[5] (b) This time, we want to write

$$\begin{bmatrix} 4 \\ 0 \\ -6 \end{bmatrix} = a \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} + c \begin{bmatrix} -6 \\ 8 \\ 1 \end{bmatrix}.$$

This results in the system of equations

$$2a - 6c = 4$$
$$4b + 8c = 0$$
$$a + 2b + c = -6.$$

From the first and second equations we have

$$a = 2 + 3c \quad \text{and} \quad b = -2c.$$

We can substitute both of these into the third equation, giving

$$(2 + 3c) + 2(-2c) + c = -6 \quad \Longrightarrow \quad 2 = -6,$$

which is impossible. Thus there are no solutions to this system of equations, and therefore no scalars a, b and c which permit us to express $\left[\begin{smallmatrix} 4 \\ 0 \\ -6 \end{smallmatrix}\right]$ as a linear combination of the given vectors.

[2] 4. (a) We have

$$\overrightarrow{AB} = \begin{bmatrix} 4-1 \\ 3-0 \\ -2-4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}.$$

[4] (b) Suppose C has coordinates (x, y, z), so that the vector

$$\overrightarrow{AC} = \begin{bmatrix} x-1 \\ y-0 \\ z-4 \end{bmatrix} = \begin{bmatrix} x-1 \\ y \\ z-4 \end{bmatrix}.$$

We want

$$\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

Thus we set

$$\begin{aligned} x-1 &= 1 \\ y &= 1 \\ z-4 &= -2 \end{aligned}$$

so $x = 2$, $y = 1$, $z = 2$. Hence C is the point $(2, 1, 2)$.

[3] 5. (a) Note that

$$\|\mathbf{u}\| = \sqrt{(-1)^2 + (-4)^2 + 8^2} = \sqrt{81} = 9.$$

Thus a unit vector in the same direction as $\mathbf{u}$ is

$$\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{9}\begin{bmatrix} -1 \\ -4 \\ 8 \end{bmatrix}.$$

[2] (b) Since we have now found a unit vector in the direction of $\mathbf{u}$, a vector of length 6 in the opposite direction to $\mathbf{u}$ is given by

$$-6 \cdot \frac{1}{9}\begin{bmatrix} -1 \\ -4 \\ 8 \end{bmatrix} = -\frac{2}{3}\begin{bmatrix} -1 \\ -4 \\ 8 \end{bmatrix}.$$

[4] 6. We set

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 0 \\ 3x(2) + x(x) + (-1)(-x) &= 0 \\ 6x + x^2 + x &= 0 \\ x^2 + 7x &= 0 \\ x(x+7) &= 0, \end{aligned}$$

so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal when $x = 0$ or $x = -7$.

[6] 7. We are given that $\|\mathbf{u}\| = 1$, $\|\mathbf{v}\| = 1$, and

$$(\mathbf{u} - 5\mathbf{v}) \cdot (\mathbf{v} - 3\mathbf{u}) = 0.$$

Expanding the latter expression yields

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} - 5\mathbf{v}\cdot\mathbf{v} - 3\mathbf{u}\cdot\mathbf{u} + 15\mathbf{v}\cdot\mathbf{u} \\ \mathbf{u}\cdot\mathbf{v} - 5\|\mathbf{v}\|^2 - 3\|\mathbf{u}\|^2 + 15\mathbf{u}\cdot\mathbf{v} &= 0 \\ 16\mathbf{u}\cdot\mathbf{v} - 5(1)^2 - 3(1)^2 &= 0 \\ \mathbf{u}\cdot\mathbf{v} &= \frac{1}{2}. \end{aligned}$$

Since

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta) = (1)(1)\cos(\theta) = \cos(\theta),$$

we now have $\cos(\theta) = \frac{1}{2}$ and so $\theta = \frac{\pi}{3}$.