MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 1.5 **Math 2050 Worksheet** WINTER 2018

SOLUTIONS

1. First we want a unit vector $\tilde{\mathbf{u}}$ in the same direction as $\mathbf{u}$. This is

$$\tilde{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{54}}\begin{bmatrix} 4 \\ -1 \\ -1 \\ 6 \end{bmatrix}$$

and so a vector in the same direction as $\mathbf{u}$ of length 4 must be

$$4\tilde{\mathbf{u}} = \frac{4}{\sqrt{54}}\begin{bmatrix} 4 \\ -1 \\ -1 \\ 6 \end{bmatrix}.$$

2. We have $\mathbf{u} \cdot \mathbf{v} = 3$, $\|\underline{u}\| = 5$ and $\|\underline{v}\| = 10$ so

$$\cos(\theta) = \frac{3}{50}$$

so $\theta \approx 1.5$ radians or 86.6 degrees.

3. (a) We let

$$k_1\mathbf{u} + k_2\mathbf{v} = k_1\begin{bmatrix} -4 \\ 1 \end{bmatrix} + k_2\begin{bmatrix} 6 \\ 7 \end{bmatrix} = 0,$$

so $-4k_1 + 6k_2 = 0$ and $k_1 + 7k_2 = 0$. From the second equation we have $k_1 = -7k_2$ and substituting this into the first equation gives

$$-4(-7k_2) + 6k_2 = 34k_2 = 0$$

so $k_2 = 0$ and hence $k_1 = 0$. Thus only the trivial combination exists and so $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

(b) We let

$$k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = k_1\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + k_3\begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = 0,$$

so $2k_1 + 3k_3 = 0$, $5k_1 - k_2 + 2k_3 = 0$ and $-k_1 - k_2 - 4k_3 = 0$. From the first equation we have $k_3 = -\frac{2}{3}k_1$. Substituting this into the second equation gives

$$5k_1 - k_2 + 2\left(-\frac{2}{3}k_1\right) = \frac{11}{3}k_1 - k_2 = 0$$

so $k_2 = \frac{11}{3}k_1$. Substituting both of these into the third equation leads to

$$-k_1 - \frac{11}{3}k_1 - 4\left(-\frac{2}{3}k_1\right) = -2k_1 = 0$$

so $k_1 = 0$, and hence $k_2 = 0$ and $k_3 = 0$ also. Thus $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

(c) We let

$$k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = k_1 \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + k_3 \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} = 0,$$

so $2k_1 + 2k_3 = 0$, $5k_1 - k_2 + 2k_3 = 0$ and $-k_1 - k_2 - 4k_3 = 0$. From the first equation we have $k_3 = -k_1$. Substituting this into the second equation gives

$$5k_1 - k_2 + 2(-k_1) = 3k_1 - k_2 = 0$$

so $k_2 = 3k_1$. Substituting both these into the third equation yields

$$-k_1 - 3k_1 - 4(-k_1) = 0,$$

which holds for any value of k_1. Hence we have an infinite number of solutions, such as $k_1 = 1$, $k_2 = 3$, $k_3 = -1$. Therefore, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

(d) We let

$$k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} + k_4\mathbf{x} = k_1 \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + k_3 \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} + k_4 \begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix} = 0,$$

so $2k_1 + 3k_3 - 4k_4 = 0$, $5k_1 - k_2 + 2k_3 = 0$ and $-k_1 - k_2 - 4k_3 + 6k_4 = 0$. From the first equation, we have $k_1 = 2k_4 - \frac{3}{2}k_3$. Substituting this into the second equation gives

$$5(2k_4 - \frac{3}{2}k_3) - k_2 + 2k_3 = 10k_4 - \frac{11}{2}k_3 - k_2 = 0$$

so $k_2 = 10k_4 - \frac{11}{2}k_3$. Substituting both of these into the third equation gives

$$-(2k_4 - \frac{3}{2}k_3) - (10k_4 - \frac{11}{2}k_3) - 4k_3 + 6k_4 = -6k_4 - 3k_3 = 0$$

so $k_3 = -\frac{1}{2}k_4$, where we have established no limitation on k_4. Thus there is an infinite number of solutions, such as $k_4 = 4$, $k_3 = -2$, $k_2 = 51$, $k_1 = 11$ and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$ are linearly dependent.

(e) We let

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + k_3\mathbf{u}_3 + k_4\mathbf{u}_4 = k_1 \begin{bmatrix} 5 \\ 0 \\ 1 \\ 3 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 5 \\ -1 \\ -2 \end{bmatrix} + k_3 \begin{bmatrix} -3 \\ 6 \\ -1 \\ 0 \end{bmatrix} + k_4 \begin{bmatrix} 2 \\ -2 \\ 7 \\ 0 \end{bmatrix} = 0$$

so $5k_1 + k_2 - 3k_3 + 2k_4 = 0$, $5k_2 + 6k_3 - 2k_4 = 0$, $k_1 - k_2 - k_3 + 7k_4 = 0$ and $3k_1 - 2k_2 = 0$. From the fourth equation we have $k_2 = \frac{3}{2}k_1$. From the third equation we have

$$k_1 - \frac{3}{2}k_1 - k_3 + 7k_4 = -\frac{1}{2}k_1 - k_3 + 7k_4 = 0$$

so $k_3 = 7k_4 - \frac{1}{2}k_1$. Substituting the value for k_2 into the second equation yields

$$5\left(\frac{3}{2}k_1\right) + 6k_3 - 2k_4 = \frac{15}{2}k_1 + 6k_3 - 2k_4 = 0$$

so $k_3 = \frac{1}{3}k_4 - \frac{5}{4}k_1$. Setting these last two equations equal to each other, we have

$$7k_4 - \frac{1}{2}k_1 = \frac{1}{3}k_4 - \frac{5}{4}k_1 \quad \Longrightarrow \quad \frac{3}{4}k_1 = -\frac{20}{3}k_4 \quad \Longrightarrow \quad k_4 = -\frac{9}{80}k_1.$$

Finally, substituting all of this into the first equation, we have

$$5k_1 + \frac{3}{2}k_1 - 3\left(-\frac{103}{80}k_1\right) + 2\left(-\frac{9}{80}k_1\right) = \frac{811}{80}k_1 = 0.$$

Thus $k_1 = 0$, and so $k_2 = 0$, $k_3 = 0$ and finally $k_4 = 0$. So $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$ are linearly independent.

4. Let $\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. We assume that $\mathbf{u}$ is orthogonal to every vector in $\mathbb{R}^n$; in particular, this means that it is orthogonal to itself so $\mathbf{u} \cdot \mathbf{u} = 0$. But then

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \cdots + x_n^2 = 0.$$

But consider any real number z. We know that $z^2 > 0$ if $z \neq 0$, and $z^2 = 0$ if and only if $z = 0$. If any $x_i \neq 0$ then the sum of the squares would have to be positive; thus it must be that $x_i = 0$ for all i. Thus we have

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}.$$

Alternatively, since $\mathbf{u}$ is orthogonal to every vector in $\mathbf{R}^n$, it must be orthogonal to the standard basis vectors. But observe that

$$\mathbf{u} \cdot \mathbf{e}_1 = x_1(1) + x_2(0) + x_3(0) + \cdots + x_n(0) = x_1,$$

so $x_1 = 0$. Similarly, for all i,

$$\mathbf{u} \cdot \mathbf{e}_i = x_i$$

so $x_i = 0$. Thus $\mathbf{u} = \mathbf{0}$.