SOLUTIONS

1. Observe that
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = 3, \quad \mathbf{u} \cdot \mathbf{u} = 42, \quad \mathbf{v} = 14.$$
Then
$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{3}{14} \begin{bmatrix} -3 \\ -1 \\ -2 \end{bmatrix} \quad \text{and} \quad \operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{3}{42} \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}.$$

2. Consider any point Q on the line, say, $Q(4,3,4)$. Then $\mathbf{u} = \overrightarrow{QP} = \begin{bmatrix} -4 \\ -4 \\ -3 \end{bmatrix}$. We want the projection $\mathbf{p}$ of $\mathbf{u}$ onto the direction vector $\mathbf{d}$ of the line:
$$\mathbf{p} = \operatorname{proj}_{\mathbf{d}} \mathbf{u} = \frac{-5}{10} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}.$$
The distance from P to ℓ will be the length of $\mathbf{u} - \mathbf{p}$. So
$$\mathbf{u} - \mathbf{p} = \frac{1}{2} \begin{bmatrix} -9 \\ -8 \\ -3 \end{bmatrix} \quad \Longrightarrow \quad \|\mathbf{u} - \mathbf{p}\| = \frac{1}{2}\sqrt{(-9)^2 + (-8)^2 + (-3)^2} = \frac{1}{2}\sqrt{154}.$$

3. (a) First we choose a point Q in the plane π, say, $Q(0,4,0)$. The vector $\mathbf{u}$ from P to Q is $\mathbf{u} = \overrightarrow{PQ} = \begin{bmatrix} 7 \\ 6 \\ 2 \end{bmatrix}$. A normal to the plane is $\mathbf{n} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$ and we project $\mathbf{u}$ onto $\mathbf{n}$ to obtain $\mathbf{p}$:
$$\mathbf{p} = \operatorname{proj}_{\mathbf{n}} \mathbf{u} = -\frac{7}{11} \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \frac{7}{11} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$
Then the distance from the point to π is
$$\|\underline{p}\| = \frac{7}{11}\sqrt{1^2 + (-1)^2 + 3^2} = \frac{7\sqrt{11}}{11}.$$

   (b) Let $R(x,y,z)$ be the point in π closest to P. Then R is the terminating point of the vector $\mathbf{p} = \overrightarrow{PR}$, and we have that
$$\frac{7}{11} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} x+7 \\ y+2 \\ z+2 \end{bmatrix}$$
so $x = -\frac{70}{11}$, $y = -\frac{29}{11}$ and $z = -\frac{1}{11}$: the point closest to P is $(-\frac{70}{11}, -\frac{29}{11}, -\frac{1}{11})$.

4. (a) Any vector in the plane will obey the equation $z = 6x - y$ so a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is of the form

$$\begin{bmatrix} x \\ y \\ 6x - y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

and so two vectors in the plane are $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. These are not orthogonal, but we can project $\mathbf{u}$ onto $\mathbf{v}$ to obtain the vector $\mathbf{p}$:

$$\mathbf{p} = \operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{-6}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{u} - \mathbf{p} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

and this is orthogonal to $\mathbf{v}$.

(b) From part (a), two orthogonal vectors in the plane are $\mathbf{e} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$. So then

$$\operatorname{proj}_{\pi} \mathbf{w} = \frac{5}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \frac{33}{19} \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = \frac{1}{38} \begin{bmatrix} 66 \\ 293 \\ 103 \end{bmatrix}.$$