MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SOLUTIONS

1. (a) Let $A = \begin{bmatrix} x \\ y \end{bmatrix}$. Then $\begin{bmatrix} 4-x \\ -4-y \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$ so $4 - x = 3$ implying $x = 1$, and $-4 - y = -7$, implying $y = 3$. Hence $A = (1, 3)$. A sketch appears below.

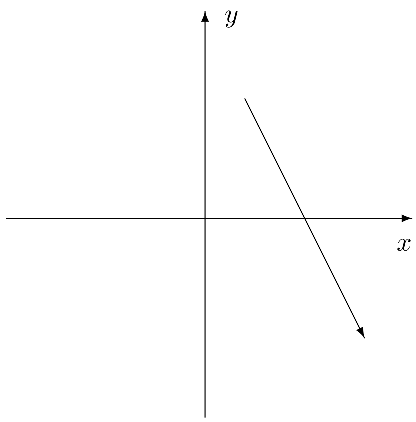

Figure 1: Question 1(a)

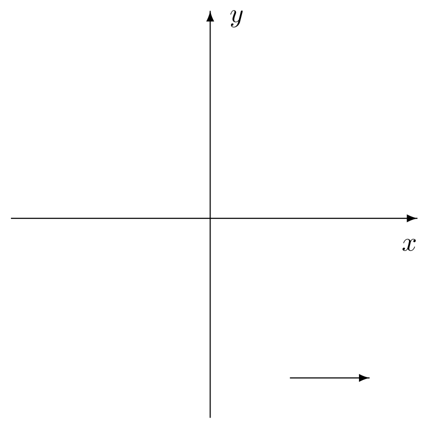

Figure 2: Question 1(b)

(b) Let $A = \begin{bmatrix} x \\ y \end{bmatrix}$. Then $\begin{bmatrix} 4-x \\ -4-y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ so $4 - x = 2$ implying $x = 2$, and $-4 - y = 0$, implying $y = -4$. Hence $A = (2, -4)$. A sketch appears above.

(c) Let $A = \begin{bmatrix} x \\ y \end{bmatrix}$. Then $\begin{bmatrix} 4-x \\ -4-y \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ so $4 - x = -1$ implying $x = 5$, and $-4 - y = -1$, implying $y = -3$. Hence $A = (5, -3)$. A sketch appears below.

(d) Let $A = \begin{bmatrix} x \\ y \end{bmatrix}$. Then $\begin{bmatrix} 4-x \\ -4-y \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$ so $4 - x = 4$ implying $x = 0$, and $-4 - y = -4$, implying $y = 0$. Hence $A = (0, 0)$. A sketch appears below.

2. (a) $\begin{bmatrix} 4 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 0 \end{bmatrix} - \begin{bmatrix} -6 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -6 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -7 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$

(b) $6\begin{bmatrix} -3 \\ 3 \\ -7 \end{bmatrix} - 4\begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -18 \\ 18 \\ -42 \end{bmatrix} + \begin{bmatrix} 0 \\ -20 \\ 4 \end{bmatrix} = \begin{bmatrix} -18 \\ -2 \\ -38 \end{bmatrix}$

(c) $-k\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} k \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4k \\ 0 \\ -k \end{bmatrix} + \begin{bmatrix} 3k \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 7k \\ 3 \\ -6-k \end{bmatrix}$

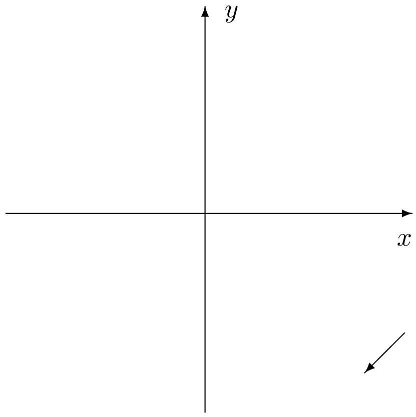

Figure 3: Question 1(c)

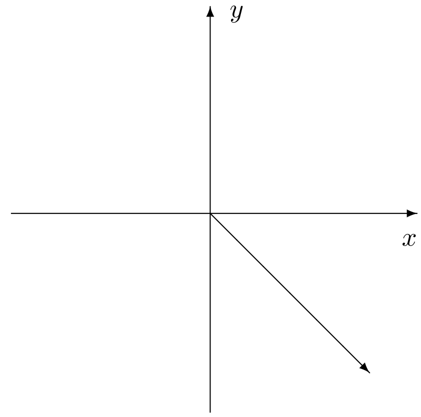

Figure 4: Question 1(d)

3. (a) We want to find scalars a and b such that

$$a\begin{bmatrix}0\\1\end{bmatrix} + b\begin{bmatrix}2\\9\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}.$$

Then we have $2b = 1$ and $a + 9b = 6$. From the first equation, we obtain $b = \frac{1}{2}$; substitution of this into the second equation gives $a = 6 - \frac{9}{2} = \frac{3}{2}$. Hence

$$\begin{bmatrix}1\\6\end{bmatrix} = \frac{3}{2}\begin{bmatrix}0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\9\end{bmatrix}.$$

(b) We want to find scalars a and b such that

$$a\begin{bmatrix}6\\4\end{bmatrix} + b\begin{bmatrix}20\\-8\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}.$$

Then we have $6a + 20b = 1$ and $4a - 8b = 6$. The second equation implies $b = \frac{1}{2}a - \frac{3}{4}$, and substitution of this into the first equation gives

$$6a + 20\left(\frac{1}{2}a - \frac{3}{4}\right) = 1 \quad\Longrightarrow\quad 16a = 16$$

so $a = 1$, and therefore $b = \frac{1}{2} - \frac{3}{4} = -\frac{1}{4}$. Hence

$$\begin{bmatrix}1\\6\end{bmatrix} = \begin{bmatrix}6\\4\end{bmatrix} - \frac{1}{4}\begin{bmatrix}20\\-8\end{bmatrix}.$$

4. (a) We want to find scalars a, b and c such that

$$a\begin{bmatrix}4\\1\\3\end{bmatrix} + b\begin{bmatrix}-1\\1\\1\end{bmatrix} + c\begin{bmatrix}6\\0\\-2\end{bmatrix} = \begin{bmatrix}-7\\0\\6\end{bmatrix}.$$

So then we have $4a - b + 6c = -7$, $a + b = 0$ and $3a + b - 2c = 6$. The second of these equations implies $b = -a$. Substituting this into the third equation yields $3a - a - 2c = 6$ so $c = a - 3$. And substitution of both of these into the first equation gives

$$4a - (-a) + 6(a - 3) = -7 \quad \Longrightarrow \quad 11a = 11$$

so $a = 1$. Hence $b = -1$ and $c = -2$. Then we can write

$$\begin{bmatrix} -7 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}.$$

(b) We want to find scalars a and b such that

$$a \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} + b \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 6 \end{bmatrix}.$$

Then we have $3a + 4b = -7$, $-4a - 3b = 0$ and $3a + 2b = 6$. From the second equation, we get $b = -\frac{4}{3}a$. Substituting this into the first equation leads to

$$3a + 4\left(-\frac{4}{3}a\right) = -7 \quad \Longrightarrow \quad -\frac{7}{3}a = -7$$

and hence $a = 3$. Thus we get $b = -\frac{4}{3}(3) = -4$. But while this satisfies the first two equations, we must also ensure that it satisfies the third: $3(3) + 2(-4) = 1 \neq 6$, and hence there is no solution. This means that no such linear combination exists.

(c) We want to find scalars a and b such that

$$a \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix} + b \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 6 \end{bmatrix}.$$

Then we have $a = -7$, $-5a + 7b = 0$, and $-3a + 3b = 6$. Substituting $a = -7$ into the second equation, we get

$$-5(-7) + 7b = 0 \quad \Longrightarrow \quad 7b = -35$$

so $b = -5$. Again, though, we need to make sure that this satisfies the last equation as well. But this time, $-3(-7) + 3(-5) = 6$, so we have shown that

$$\begin{bmatrix} -7 \\ 0 \\ 6 \end{bmatrix} = -7 \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix}.$$

5. (a) We are given that $a\mathbf{u}+b\mathbf{v}=\mathbf{w}$ for certain scalars a and b. We wish to find scalars c and d such that

$$c(5\mathbf{u})+d(-\mathbf{v})=\mathbf{w}.$$

But setting $a=5c$ and $b=-d$, we see that $c=\frac{1}{5}a$ and $d=-b$ are precisely the scalars needed. Hence $\mathbf{w}$ is a linear combination of $5\mathbf{u}$ and $-\mathbf{v}$.

(b) We are given that $a\mathbf{u}+b\mathbf{v}=\mathbf{w}$ for certain scalars a and b. We wish to find scalars c and d such that

$$c(k\mathbf{u})+d(\ell\mathbf{v})=\mathbf{w}.$$

But setting $a=ck$ and $b=d\ell$, we see that $c=\frac{1}{k}a$ and $d=\frac{1}{\ell}b$ are precisely the scalars needed (and both of these scalars exist because both k and ℓ are non-zero). Hence $\mathbf{w}$ is a linear combination of $k\mathbf{u}$ and $\ell\mathbf{v}$.

6. (a) We assume that $\mathbf{u}$ and $\mathbf{v}$ are parallel, which means that there exists a non-zero scalar k such that $\mathbf{u}=k\mathbf{v}$. We want to find scalars a and b (not equal to zero) such that

$$a\mathbf{u}+b\mathbf{v}=\mathbf{0}.$$

But then

$$a\mathbf{u}+b\mathbf{v}=a(k\mathbf{v})+b\mathbf{v}=(ak+b)\mathbf{v}$$

and if $ak+b=0$ then we will have $a\mathbf{u}+b\mathbf{v}=\mathbf{0}$. Hence we can choose $b=-ak$ and a to be any non-zero real number; then a and b are the desired non-zero scalars.

(b) We assume that there exist scalars a and b (not equal to zero) such that

$$a\mathbf{u}+b\mathbf{v}=\mathbf{0}.$$

We wish to show that $\mathbf{u}=k\mathbf{v}$ for some non-zero scalar k. But we see that

$$a\mathbf{u}=-b\mathbf{v} \quad\Longrightarrow\quad \mathbf{u}=-\frac{b}{a}\mathbf{v},$$

so we can let $k=-\frac{b}{a}$ (which must be defined, since $a\neq 0$), and hence $\mathbf{u}=k\mathbf{v}$, that is, $\mathbf{u}$ and $\mathbf{v}$ are parallel.