Section 3.1: The Determinant of a Matrix

For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ its <u>determinant</u> is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For a 1×1 matrix $[a]$ then its determinant is $|a| = a$.

We want to define the determinant of any square matrix.

<u>Def'n</u>: If A is a square matrix, the <u>(i,j)-minor</u> of A is the determinant of the matrix obtained by deleting row i and column j from A. It is denoted by M_{ij}.

eg If $A = \begin{bmatrix} 3 & 7 & -1 \\ 2 & 4 & 0 \\ -3 & -3 & -1 \end{bmatrix}$ then $M_{11} = \begin{vmatrix} 4 & 0 \\ -3 & -1 \end{vmatrix} = -4$

$$M_{12} = \begin{vmatrix} 2 & 0 \\ -3 & -1 \end{vmatrix} = -2$$

$$M_{23} = \begin{vmatrix} 3 & 7 \\ -3 & -3 \end{vmatrix} = 12$$

We can then assemble the <u>matrix of minors</u>:

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -2 & 6 \\ -10 & -6 & 12 \\ 4 & 2 & -2 \end{bmatrix}$$

Def'n: The (i,j)-cofactor of a square matrix A is given by $(-1)^{i+j} M_{ij}$. Thus it is M_{ij} if $i+j$ is even, and $-M_{ij}$ if $i+j$ is odd. It is denoted by C_{ij}.

eg For $A = \begin{bmatrix} 3 & 7 & -1 \\ 2 & 4 & 0 \\ -3 & -3 & -1 \end{bmatrix}$, we have $M_{11} = -4$ so $C_{11} = -4$

$M_{12} = -2$ so $C_{12} = 2$

$M_{23} = 12$ so $C_{23} = -12$

The matrix of cofactors is

$$C = \begin{bmatrix} -4 & 2 & 6 \\ 10 & -6 & -12 \\ 4 & -2 & -2 \end{bmatrix}.$$

In general, to obtain the matrix of cofactors from the matrix of minors, we just multiply each entry in the following pattern:

$$\begin{matrix} + & - & + & - & + & - & \cdots \\ - & + & - & + & - & + & \cdots \\ + & - & + & - & + & - & \cdots \\ - & + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

Theorem: Let C be the matrix of cofactors for a square matrix A. Then A commutes with C^T and $AC^T = C^TA = kI$ for some scalar k. This value of k is the determinant of A, denoted by $|A|$ or $\det(A)$.

eg For $A = \begin{bmatrix} 3 & 7 & -1 \\ 2 & 4 & 0 \\ -3 & -3 & -1 \end{bmatrix}$ we have $C^T = \begin{bmatrix} -4 & 10 & 4 \\ 2 & -6 & -2 \\ 6 & -12 & -2 \end{bmatrix}$

so

$$AC^T = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} = -4I$$

Hence $\boxed{\det(A) = -4}$.

In practice to find a determinant we just need to multiply the ith row of A by the ith column of C^T, which is the ith row of C.

This is the Laplace expansion: we choose any row of A and we multiply each entry of the row by the corresponding cofactor.

eg $A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -3 & 4 \\ 1 & 0 & -2 \end{bmatrix}$

We can expand along the 1st row:

$$\det(A) = 3 \begin{vmatrix} -3 & 4 \\ 0 & -2 \end{vmatrix} + 1 \cdot (-1) \begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} + 0$$

$$= 3(6-0) - (4-4)$$

$$\boxed{= 18}$$

eg $A = \begin{bmatrix} 1 & -1 & 2 \\ 7 & 3 & 1 \\ 2 & -3 & -4 \end{bmatrix}$

We could expand along any row (or column) because there are no zeros. If we choose the 1st row we get

$$\det(A) = \begin{vmatrix} 3 & 1 \\ -3 & -4 \end{vmatrix} - (-1)\begin{vmatrix} 7 & 1 \\ 2 & -4 \end{vmatrix} + 2\begin{vmatrix} 7 & 3 \\ 2 & -3 \end{vmatrix}$$

$$= (-9) + (-30) + 2(-27)$$

$$\boxed{= -93}$$

But we could use, say, the 2nd row:

$$\det(A) = -7\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}$$

$$= -7 \cdot 10 + 3(-8) - (-1)$$

$$\boxed{= -93}$$

eg $A = \begin{bmatrix} 3 & -1 & -2 & 5 \\ 1 & 7 & -6 & 0 \\ 4 & 0 & 0 & -2 \\ -6 & 1 & -1 & 3 \end{bmatrix}$

We expand along the 3rd row:

$$\det(A) = 4\begin{vmatrix} -1 & -2 & 5 \\ 7 & -6 & 0 \\ 1 & -1 & 3 \end{vmatrix} - 0 + 0 - (-2)\begin{vmatrix} 3 & -1 & -2 \\ 1 & 7 & -6 \\ -6 & 1 & -1 \end{vmatrix}$$

Here $\begin{vmatrix} -1 & -2 & 5 \\ 7 & -6 & 0 \\ 1 & -1 & 3 \end{vmatrix} = -7\begin{vmatrix} -2 & 5 \\ -1 & 3 \end{vmatrix} + (-6)\begin{vmatrix} -1 & 5 \\ 1 & 3 \end{vmatrix} - 0$

$$= -7(-1) - 6(-8)$$

$$= 55$$

and $\begin{vmatrix} 3 & -1 & -2 \\ 1 & 7 & -6 \\ -6 & 1 & -1 \end{vmatrix} = 3\begin{vmatrix} 7 & -6 \\ 1 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -6 \\ -6 & -1 \end{vmatrix} + (-2)\begin{vmatrix} 1 & 7 \\ -6 & 1 \end{vmatrix}$

$$= 3(-1) + (-37) - 2(43)$$

$$= -126$$

Thus $\det(A) = 4(55) + 2(-126) \boxed{= -32}$.

Theorem: Let A be a square matrix and C be its matrix of cofactors. If $\det(A) \neq 0$ then A is invertible and $A^{-1} = \frac{1}{\det(A)} C^T$.

Proof: We have shown that $AC^T = (\det A) I$ so, if $\det(A) \neq 0$ then

$$\frac{1}{\det(A)} AC^T = I \quad \text{so} \quad A\left(\frac{1}{\det(A)} C^T\right) = I.$$

Thus $A^{-1} = \frac{1}{\det(A)} C^T$.

eg For $A = \begin{bmatrix} 3 & 7 & -1 \\ 2 & 4 & 0 \\ -3 & -3 & -1 \end{bmatrix}$ we have found $\det(A) = -4$

and $C = \begin{bmatrix} -4 & 2 & 6 \\ 10 & -6 & -12 \\ 4 & -2 & -2 \end{bmatrix}$ so A is invertible and

$$A^{-1} = \frac{1}{-4} \begin{bmatrix} -4 & 10 & 4 \\ 2 & -6 & -2 \\ 6 & -12 & -2 \end{bmatrix}.$$

The matrix C^T is also called the adjoint of A, and is denoted by $\mathrm{adj}(A)$. Thus we can also write

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A) \quad \text{if } \det(A) \neq 0.$$