Section 2.6: LU Factorisation

eg Let $A = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 2 & 5 & -3 & 6 \\ -4 & -20 & 1 & 1 \end{bmatrix}$. Represent the reduction of A to row-echelon form using elementary matrices.

$$\begin{bmatrix} 1 & 4 & 0 & 0 \\ 2 & 5 & -3 & 6 \\ -4 & -20 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - (-4)R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & -3 & -3 & 6 \\ 0 & -4 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to (-\frac{1}{3})R_2} \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & -4 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - (-4)R_2} \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 5 & -7 \end{bmatrix} = U$$

The corresponding elementary matrices are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Thus $U = E_4 E_3 E_2 E_1 A$

This also means that $A = (E_4 E_3 E_2 E_1)^{-1} U$

$$= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ -4 & -4 & 1 \end{bmatrix} U$$

$$= LU$$

This is an <u>LU-factorisation</u> of A.

<u>Def'n</u>: An LU-factorisation of a matrix A writes the matrix as $A = LU$ where L is a square lower triangular matrix and U is a row-echelon matrix of the same size as A.

If we bring A to row-echelon form using only the 3rd elementary row operation ($R_j \to R_j - kR_i$) then the diagonal entries of L will all be one, and the (j,i)-entry will be k.

eg Find an LU-factorisation of $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - (-\frac{1}{2})R_1]{R_2 \to R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - (-2)R_2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix} = U$$

Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$.

If we want to solve an equation $A\underline{x} = \underline{b}$ then we can now write this as $LU\underline{x} = \underline{b}$. If we let $\underline{y} = U\underline{x}$ then we could first solve $L\underline{y} = \underline{b}$ and then solve $U\underline{x} = \underline{y}$.

eg Use LU-factorisation to solve the system

$$\begin{cases} 2x + 4y + 2z = -6 \\ x + y + 2z = 2 \\ -x \qquad + 2z = 3 \end{cases}$$

Here $L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}$ so if we set $L\underline{y} = \underline{b}$ then

we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 3 \end{bmatrix}$$

We use forward substitution:

$$u = -6$$

$$\tfrac{1}{2}u + v = 2 \to v = 2 + 3 = 5$$

$$-\tfrac{1}{2}u - 2v + w = 3 \to w = -3 + 10 + 3 = 10$$

Now we have $U\underline{x} = y$ so

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \\ 10 \end{bmatrix}$$

We use back substitution:

$$5z = 10 \rightarrow z = 2$$

$$-y + z = 5 \rightarrow y = 2 - 5 = -3$$

$$2x + 4y + 2z = -6 \rightarrow 2x = -6 + 12 - 4$$

$$x = 1$$