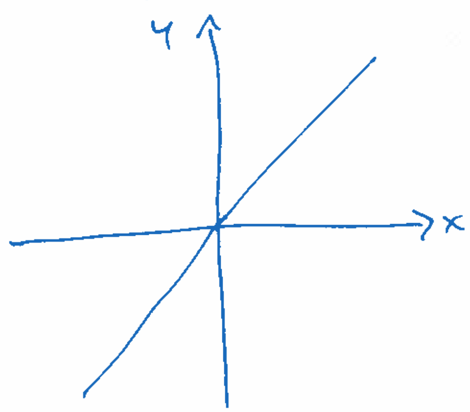
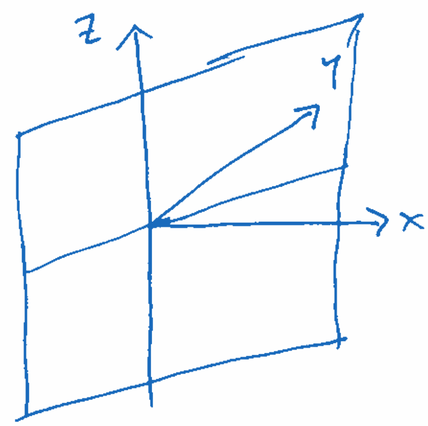
Section 13: Lines, Planes, Cross Product

In 2 dimensions the graph of a linear equation $ax+by=c$ is a line in the xy-plane.





In 3 dimensions, however, $ax+by=c$ is the equation of a plane. Specifically, it is a plane that is orthogonal to the xy-plane.

We define a <u>normal</u> to a plane to be any non-zero vector which is orthogonal to that plane.

<u>Theorem</u>: A plane π in 3-space with normal $\underline{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the graph of the equation $ax+by+cz=d$ where d is a constant.

<u>Proof</u>: The normal $\underline{n}$ is orthogonal to the plane, and thus $\underline{n}$ is orthogonal to every vector which lies in the plane.

Assume that $P_0(x_0, y_0, z_0)$ is a point in the plane.

Let $P(x,y,z)$ be a second point in π.

Then $\overrightarrow{P_0P} = \begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix}$ is a vector in Π, and so

is orthogonal to $\underline{n}$.

Thus $\overrightarrow{P_0P} \cdot \underline{n} = 0$

$$\begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{d}$$

eg Find the equation of the plane through the point $(1,-1,3)$ with normal $\underline{n} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

This plane has an equation of the form $3x - y + 2z = d$.

Since $(1,-1,3)$ lies in the plane,

$$3 \cdot 1 - (-1) + 2 \cdot 3 = d$$

$$10 = d$$

The equation of the plane is $\boxed{3x - y + 2z = 10}$.

eg Find an equation of the plane through $(3,-1,2)$ that is parallel to the plane with equation $2x - 3y = 6$.

A normal to the plane with equation $2x-3y=6$ is

$$\underline{n} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$$ and it must also be a normal to the

given plane because these two planes are parallel.

Its equation has the form $2x-3y=d$

$$2\cdot 3 - 3(-1) = d$$

$$9 = d$$

so the equation of this plane is $\boxed{2x-3y=9}$.

Given any two vectors that are not parallel, the set of all linear combinations of those vectors defines a plane. We say that these two vectors <u>span</u> the plane, and that the plane is <u>spanned</u> by the two vectors.

eg Consider the plane $6x-y-2z=0$. Any vector in the plane must satisfy this equation, so if $\underline{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then $y = 6x-2z$. Thus

$$\underline{u} = \begin{bmatrix} x \\ 6x-2z \\ z \end{bmatrix} = \begin{bmatrix} x \\ 6x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2z \\ z \end{bmatrix}$$

$$= x \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Thus the vectors $\begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ span the plane.

Equivalently, we could write $2z = 6x - y$

$$z = 3x - \frac{1}{2}y$$

and so

$$\underline{u} = \begin{bmatrix} x \\ y \\ 3x - \frac{1}{2}y \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 3x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ -\frac{1}{2}y \end{bmatrix}$$

$$= x \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1/2 \end{bmatrix}$$

So $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1/2 \end{bmatrix}$ also span the plane.

Now we want to take any two non-parallel vectors and use them to determine the equation of the plane that they span.

__Def'n__: Given numbers a_1, a_2, b_1, b_2 the __2x2 determinant__ is

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 .$$

eg $\begin{vmatrix} 4 & 1 \\ -2 & 7 \end{vmatrix} = 4 \cdot 7 - 1 \cdot (-2) \boxed{= 30}$

Def'n: The cross product of vectors $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

and $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is the vector

$$\underline{u} \times \underline{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \underline{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \underline{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \underline{k}$$

In Unit 3 we will see that $\underline{u} \times \underline{v}$ can also be written as the 3×3 determinant

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

eg Find $\underline{u} \times \underline{v}$ given $\underline{u} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$.

$$\underline{u} \times \underline{v} = \begin{vmatrix} -1 & 4 \\ 3 & 7 \end{vmatrix} \underline{i} - \begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} \underline{j} + \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} \underline{k}$$

$$= (-7-12)\underline{i} - (14-4)\underline{j} + (6-(-1))\underline{k}$$

$$= (-19)\underline{i} - 10\underline{j} + 7\underline{k}$$

$$\boxed{= \begin{bmatrix} -19 \\ -10 \\ 7 \end{bmatrix}}$$

Theorem: Properties of the Cross Product

If $\underline{u}, \underline{v}, \underline{w}$ are 3-dimensional vectors and k is a scalar then

1. Anticommutativity: $\underline{u} \times \underline{v} = -(\underline{v} \times \underline{u})$
2. $\underline{u} \times (\underline{v} + \underline{w}) = (\underline{u} \times \underline{v}) + (\underline{u} \times \underline{w})$
3. $(k\underline{u}) \times \underline{v} = \underline{u} \times (k\underline{v}) = k(\underline{u} \times \underline{v})$
4. $\underline{u} \times \underline{0} = \underline{0} \times \underline{u} = \underline{0}$
5. $\underline{u} \times \underline{u} = \underline{0}$
6. $\underline{u} \times \underline{v} = \underline{0}$ if and only if $\underline{u}$ and $\underline{v}$ are parallel (so $\underline{v} = k\underline{u}$)
7. $\|\underline{u} \times \underline{v}\| = \|\underline{u}\| \|\underline{v}\| \sin(\theta)$ where θ is the angle between $\underline{u}$ and $\underline{v}$

Note that the cross-product is neither commutative nor associative:

$$(\underline{u} \times \underline{v}) \times \underline{w} \neq \underline{u} \times (\underline{v} \times \underline{w})$$

eg Suppose $\underline{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\underline{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\underline{u} \times \underline{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{so} \quad (\underline{u} \times \underline{v}) \times \underline{w} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{v} \times \underline{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{so} \quad \underline{u} \times (\underline{v} \times \underline{w}) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

All of the standard basis vectors can be obtained by taking cross products of each other : $\underline{k} = \underline{i} \times \underline{j}$, $\underline{i} = \underline{j} \times \underline{k}$ and $\underline{j} = \underline{k} \times \underline{i}$.

<u>Theorem</u> : The vector $\underline{u} \times \underline{v}$ is orthogonal to both $\underline{u}$ and $\underline{v}$.

<u>Proof</u> : Let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ so

$$\underline{u} \times \underline{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

$$\underline{u} \cdot (\underline{u} \times \underline{v}) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

$$= u_1 (u_2 v_3 - u_3 v_2) + u_2 (u_3 v_1 - u_1 v_3) + u_3 (u_1 v_2 - u_2 v_1)$$

$$= u_1 u_2 v_3 - u_1 u_3 v_2 + u_2 u_3 v_1 - u_1 u_2 v_3 + u_1 u_3 v_2 - u_2 u_3 v_1$$

$= 0$ so $\underline{u}$ and $\underline{u} \times \underline{v}$ are orthogonal

Likewise $\underline{v} \cdot (\underline{u} \times \underline{v}) = 0$ so $\underline{v}$ and $\underline{u} \times \underline{v}$ are orthogonal.

eg For $\underline{u} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$ we have already found that $\underline{u} \times \underline{v} = \begin{bmatrix} -19 \\ -10 \\ 7 \end{bmatrix}$

Thus $\underline{u} \cdot (\underline{u} \times \underline{v}) = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -19 \\ -10 \\ 7 \end{bmatrix} = -38 + 10 + 28 = 0$

$\underline{v} \cdot (\underline{u} \times \underline{v}) = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -19 \\ -10 \\ 7 \end{bmatrix} = -19 - 30 + 49 = 0$

Given two vectors $\underline{u}$ and $\underline{v}$ which lie in a plane Π, a normal to Π must be $\underline{u} \times \underline{v}$.

eg Find the equation of the plane through the points $P(1,3,-2)$, $Q(1,1,5)$ and $R(2,-2,3)$.

First we need two vectors which lie in the plane and are not parallel. We might use $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, where

$$\overrightarrow{PQ} = \begin{bmatrix} 1-1 \\ 1-3 \\ 5-(-2) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 7 \end{bmatrix}$$

$$\overrightarrow{PR} = \begin{bmatrix} 2-1 \\ -2-3 \\ 3-(-2) \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}$$

Thus $\underline{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 25 \\ 7 \\ 2 \end{bmatrix}$

The equation of the plane has the form $25x + 7y + 2z = d$

Using $P(1,3,-2)$ this becomes $25 \cdot 1 + 7 \cdot 3 + 2(-2) = d$

$42 = d$

The equation of the plane is $\boxed{25x + 7y + 2z = 42}$.

Consider a line ℓ oriented in a direction $\underline{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ which passes through a point $P_0(x_0, y_0, z_0)$. Let $P(x, y, z)$ be any other point on ℓ. Then $\overrightarrow{P_0P} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$

is a vector in either the same direction as $\underline{d}$ or the opposite direction to $\underline{d}$. Thus $\underline{d}$ and $\overrightarrow{P_0P}$ are parallel so $\overrightarrow{P_0P} = t\underline{d}$ for some scalar t. Hence

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

This is the <u>vector equation</u> of a line.

The resulting equations
$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

are the <u>parametric equations</u> of a line.

eg Find the vector and parametric equations of the line through the points $P(0, 6, 2)$ and $Q(-4, 4, 3)$.

The direction vector $\underline{d}$ is parallel to $\overrightarrow{PQ}$ so

$$\underline{d} = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$$

One vector equation for this line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 2 \end{bmatrix} + t \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$$

with parametric equations
$$x = -4t$$
$$y = 6 - 2t$$
$$z = 2 + t$$

But there are many others:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 2 \end{bmatrix} + t \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix} + t \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ 4 \end{bmatrix} + s \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix} \quad \text{where } t = s+1$$

eg Determine whether $P(4,3,0)$ lies on the line with equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix} + t \begin{bmatrix} 5 \\ -3 \\ -4 \end{bmatrix}.$$

If P lies on the line, there is a scalar t for which

$x = -1 + 5t = 4 \rightarrow 5t = 5 \rightarrow t = 1$

$y = 6 - 3t = 3 \rightarrow$ For $t=1$, $y = 6 - 3 = 3$

$z = 3 - 4t = 0 \qquad z = 3 - 4 = -1 \neq 0$

Thus P does not lie on the line.

eg Find the vector equation of the line through $P(2,0,-2)$ that is parallel to the line with equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 7 \\ 7 \end{bmatrix}.$$

Since the lines are parallel, $\underline{d} = \begin{bmatrix} 3 \\ 7 \\ 7 \end{bmatrix}$ is also a direction vector for the line through P. So its equation is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 7 \\ 7 \end{bmatrix}.$$

eg Find the point of intersection of the lines with equations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ 14 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}.$$

or show that they do not intersect.

We want to determine if there are values α and β for which

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ 14 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$$

Thus we must have

$$\begin{aligned} 2 + 4\alpha &= -8 - 2\beta \\ -1 - \alpha &= 1 + \beta \\ -3\alpha &= 14 - 5\beta \end{aligned}$$

From the second equation, $\beta = -2 - \alpha$

Substituting into the third equation gives

$$-3\alpha = 14 - 5(-2-\alpha)$$

$$-3\alpha = 14 + 10 + 5\alpha$$

$$-8\alpha = 24$$

$$\alpha = -3 \rightarrow \beta = -2-(-3) = 1$$

This makes the first equation

$$2 + 4(-3) = -8 - 2(1)$$

$$-10 = -10$$

Since all 3 equations are satisfied, the lines must intersect, and they do so when

$$x = 2 + (-3)4 = -10$$

$$y = -1 + (-3)(-1) = 2$$

$$z = 0 + (-3)(-3) = 9$$

that is, at the point $(-10, 2, 9)$.

eg Determine whether the line with equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 11 \end{bmatrix} + t \begin{bmatrix} -4 \\ 9 \\ 2 \end{bmatrix}$$

intersects with the plane described by the equation $5x + 2y + z = -4$. If so, find the intersection point.

The line and the plane will intersect unless they are parallel, which would require the direction vector $\underline{d}$ of the line to be orthogonal to the normal vector $\underline{n}$ of the plane.

Here $\underline{d} = \begin{bmatrix} -4 \\ 9 \\ 2 \end{bmatrix}$ and $\underline{n} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$ so we need to determine if $\underline{d} \cdot \underline{n} = 0$.

Thus $\underline{d} \cdot \underline{n} = \begin{bmatrix} -4 \\ 9 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = -20 + 18 + 2 = 0$

Since $\underline{d}$ and $\underline{n}$ are orthogonal, the line and the plane have no point of intersection.