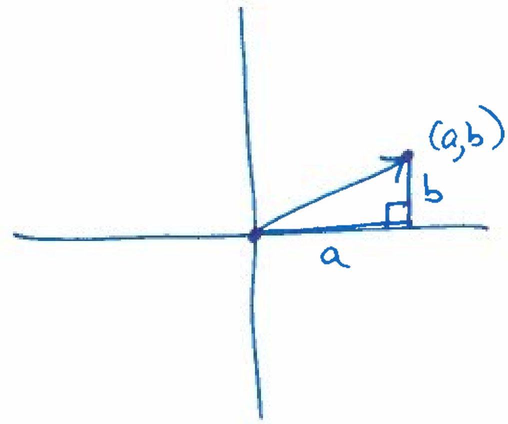
Section 1.2: Length and Direction

Consider a vector $\underline{u} = \begin{bmatrix} a \\ b \end{bmatrix}$.



It can be drawn as the hypotenuse of a right triangle with sides a and b, by the Pythagorean theorem the length of $\underline{u}$ is $\sqrt{a^2+b^2}$.

We call this the <u>norm</u> of $\underline{u}$, and denote it by $\|\underline{u}\|$.

eg The norm of $\underline{u} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$ is

$$\|\underline{u}\| = \sqrt{(-6)^2 + 8^2} = \sqrt{36+64} = \sqrt{100} \boxed{= 10}$$

Likewise, if $\underline{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then $\|\underline{u}\| = \sqrt{a^2+b^2+c^2}$.

A <u>unit vector</u> is a vector whose norm is 1. The standard basis vectors are all unit vectors, but there are many others.

eg Determine whether $\underline{u} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$ is a unit vector.

$$\|\underline{u}\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}}$$

$$= \sqrt{\frac{25}{25}}$$

$$= \sqrt{1} = 1$$

Thus $\underline{u}$ is a unit vector.

Theorem: Given a vector $\underline{u}$ and a scalar k,

$$\|k\underline{u}\| = |k| \cdot \|\underline{u}\|.$$

Proof: Let $\underline{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ so $k\underline{u} = \begin{bmatrix} ka \\ kb \end{bmatrix}$. Then

$$\begin{aligned}
\|k\underline{u}\| &= \sqrt{(ka)^2 + (kb)^2} \\
&= \sqrt{k^2a^2 + k^2b^2} \\
&= \sqrt{k^2(a^2+b^2)} \\
&= \sqrt{k^2} \cdot \sqrt{a^2+b^2} = |k| \cdot \|\underline{u}\|
\end{aligned}$$

The same process would apply to a 3-dimensional vector.

Now, given any vector $\underline{u}$, a unit vector $\underline{\hat{u}}$ which points in the same direction as $\underline{u}$ is

$$\underline{\hat{u}} = \frac{\underline{u}}{\|\underline{u}\|}$$

because $\|\underline{\hat{u}}\| = \left\| \frac{\underline{u}}{\|\underline{u}\|} \right\| = \left\| \frac{1}{\|\underline{u}\|} \underline{u} \right\| = \left| \frac{1}{\|\underline{u}\|} \right| \cdot \|\underline{u}\|$

$$= \frac{\|\underline{u}\|}{\|\underline{u}\|} = 1$$

eg Given $\underline{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we have $\|\underline{u}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$

A unit vector in the same direction as $\underline{u}$ is $\underline{\hat{u}} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

One way to multiply two vectors is the <u>dot product</u> :

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz$$

eg $$\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = 1\cdot 4 + 2\cdot 0 + (-5)\cdot 3 = 4-15 \boxed{=-11}$$

<u>Theorem</u> : <u>Properties of the Dot Product</u>

Let $\underline{u}, \underline{v}, \underline{w}$ be vectors and k be a scalar then

1. Commutativity : $\underline{u}\cdot\underline{v} = \underline{v}\cdot\underline{u}$

2. Distributivity : $\underline{u}\cdot(\underline{v}+\underline{w}) = \underline{u}\cdot\underline{v} + \underline{u}\cdot\underline{w}$

   $(\underline{u}+\underline{v})\cdot\underline{w} = \underline{u}\cdot\underline{w} + \underline{v}\cdot\underline{w}$

3. Scalar associativity : $(k\underline{u})\cdot\underline{v} = \underline{u}\cdot(k\underline{v})$

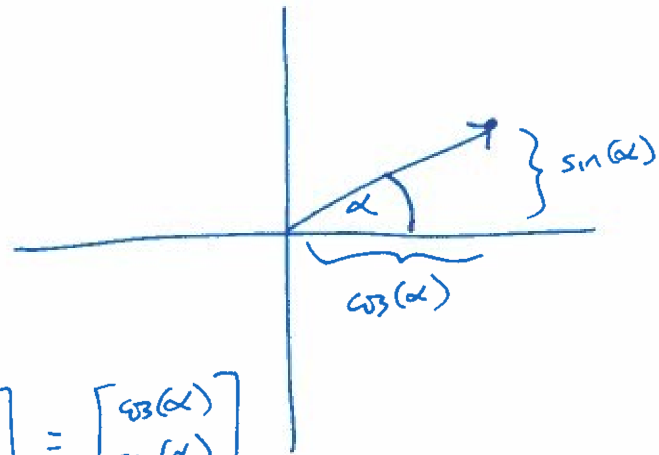
Observe that $\underline{u}\cdot\underline{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a^2 + b^2 + c^2 = \|\underline{u}\|^2$

Thus we can also define $\|\underline{u}\| = \sqrt{\underline{u}\cdot\underline{u}}$.

Note that $\underline{u}\cdot\underline{u} \geq 0$, and $\underline{u}\cdot\underline{u} = 0$ if and only if $\underline{u} = \underline{0}$.

Consider a unit vector drawn starting at the origin. Then the terminal point for the vector lies on the unit circle. We can then measure the angle from 0 to π that the vector makes with the positive x-axis.

Thus, if this angle is α, we can write a unit vector as $\begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}$.



Consider two unit vectors $\underline{u} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}$

$$\underline{v} = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix}$$

If $\beta > \alpha$ then the angle between $\underline{u}$ and $\underline{v}$ is $\theta = \beta - \alpha$ and so

$$\begin{aligned} \cos(\theta) &= \cos(\beta - \alpha) \\ &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ &= ac + bd \\ &= \underline{u} \cdot \underline{v} \end{aligned}$$

More generally, if $\underline{u}$ and $\underline{v}$ are not unit vectors then we can repeat this process with $\frac{\underline{u}}{\|\underline{u}\|}$ and $\frac{\underline{v}}{\|\underline{v}\|}$. Then we obtain

$$\frac{\underline{u}}{\|\underline{u}\|} \cdot \frac{\underline{v}}{\|\underline{v}\|} = \cos(\theta)$$

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \, \|\underline{v}\| \cos(\theta).$$

eg Find the angle between $\underline{u} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We have $\|\underline{u}\| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\|\underline{v}\| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$

$\underline{u} \cdot \underline{v} = \sqrt{3} \cdot 0 + 1 \cdot 1 = 1$

Thus $1 = 2 \cdot 1 \cdot \cos(\theta)$

$\cos(\theta) = \frac{1}{2} \rightarrow \boxed{\theta = \frac{\pi}{3}}$

<u>Def'n</u>: Vectors $\underline{u}$ and $\underline{v}$ are <u>orthogonal</u> if and only if

$\underline{u} \cdot \underline{v} = 0$.

This means that the zero vector $\underline{0}$ is orthogonal to every vector, including itself.

Likewise, the standard basis vectors are orthogonal to each other.

Finally, recall that $|\cos(\theta)| \leq 1$, so if $\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos(\theta)$ then we have the following.

<u>Theorem</u>: <u>Cauchy-Schwarz Inequality</u>

Given vectors $\underline{u}$ and $\underline{v}$, $|\underline{u} \cdot \underline{v}| \leq \|\underline{u}\| \|\underline{v}\|$.