Section 1.1: Vectors

<u>Vectors</u> are quantities which can be described by the combination of a magnitude and a direction.

<u>Scalars</u> are quantities which are described by a magnitude only.

We will begin by considering <u>two-dimensional vectors</u>. They are characterised by two <u>components</u>, written in a column and surrounded by brackets.

eg $\begin{bmatrix} 4 \\ -9 \end{bmatrix}$ or $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$ or $\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$

A vector is typically labelled by boldfacing, such as $\mathbf{u}$.

In practice, however, we write $\underline{u}$ or $\vec{u}$.

Given a vector $\underline{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ we say it is <u>equal</u> to a vector $\underline{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ if both $a=c$ and $b=d$.

eg Find all values of x and y such that $\underline{u} = \begin{bmatrix} 9x^2 \\ y-4 \end{bmatrix}$ is equal to $\underline{v} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

We set $9x^2 = 4$ and $y-4=0$

$x^2 = \frac{4}{9}$

$\boxed{y=4}$

$\boxed{x = \pm\sqrt{\frac{4}{9}} = \pm\frac{2}{3}}$

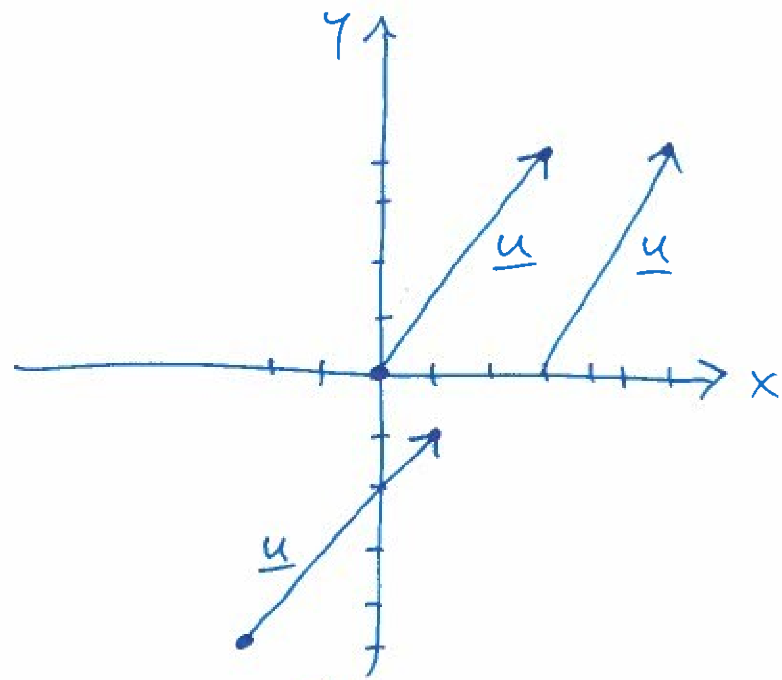
A two-dimensional vector can be represented graphically in the xy-plane by first choosing any point A with coordinates (x_0, y_0). Then we choose a second point B with coordinates (x_1, y_1) such that

$$x_1 = x_0 + a \quad \text{and} \quad y_1 = y_0 + b.$$

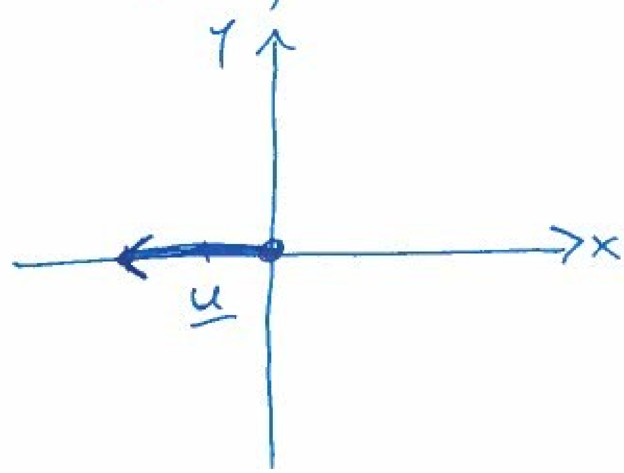
Then we can draw an arrow starting at A and ending at B, and this represents the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ where $a = x_1 - x_0$, $b = y_1 - y_0$.

But any arrow of the same magnitude and direction would still represent this vector, so there are in fact an infinite number of correct ways to graphically represent a given vector.

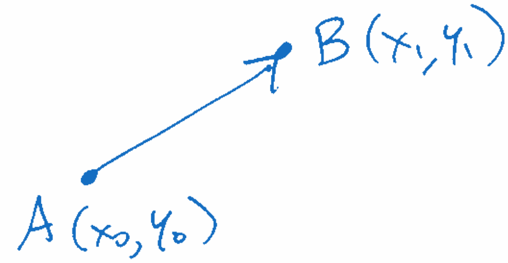
eg $\underline{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$



eg $\underline{u} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$



$$\underline{u} = \begin{bmatrix} a \\ b \end{bmatrix}$$



Given two points $A(x_0, y_0)$ and $B(x_1, y_1)$ then the vector $\overrightarrow{AB}$ that starts at A and ends at B is given by

$$\overrightarrow{AB} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix}.$$

<u>Scalar multiplication</u> is the multiplication of a vector $\underline{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ by a scalar k:

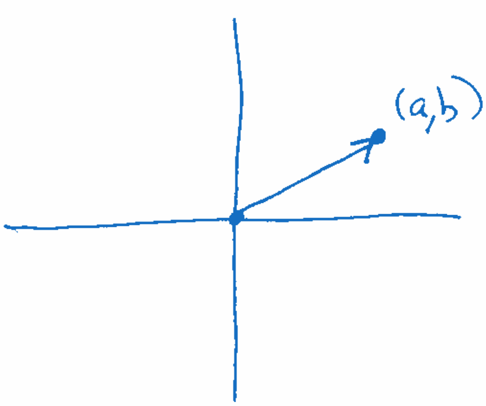
$$k\underline{u} = k\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ka \\ kb \end{bmatrix}.$$

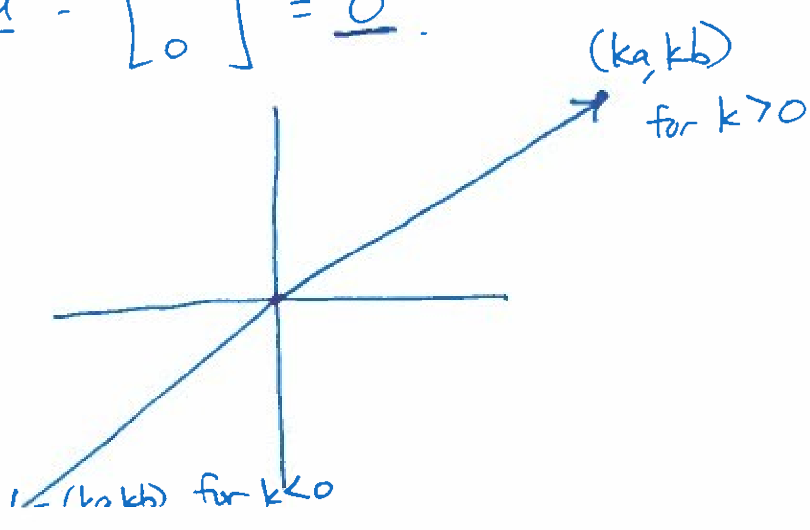
eg $$6\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -12 \\ 30 \end{bmatrix}$$

Multiplication by the scalar -1 produces the <u>negative</u> of the vector, denoted by $(-1)\underline{u} = -\underline{u}$.

Multiplication by the scalar 0 produces the <u>zero vector</u>:

$$0\underline{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underline{0}.$$

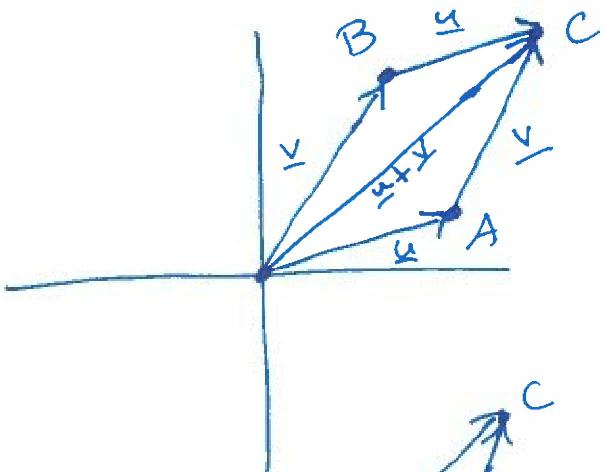




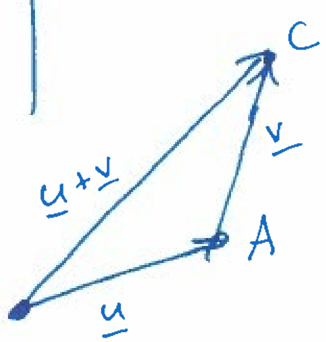
Def'n: Vectors $\underline{u}$ and $\underline{v}$ are parallel if one is a scalar multiple of the other, that is, if $\underline{v} = k\underline{u}$ or $\underline{u} = k\underline{v}$ for some scalar k.

Vector addition is the sum of two vectors where we add corresponding components:

$$\underline{u} + \underline{v} = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$$



Graphically, $\underline{u}+\underline{v}$ is the diagonal of a parallelogram with sides $\underline{u}$ and $\underline{v}$. This is the Parallelogram Rule.



Alternatively we consider $\underline{u}$ drawn such that $\underline{v}$ begins at its endpoint. Then $\underline{u}+\underline{v}$ is the vector that starts where $\underline{u}$ starts and ends where $\underline{v}$ ends. This is the Triangle Rule.

Theorem: Properties of Scalar Multiplication and Vector Addition

Let $\underline{u}, \underline{v}, \underline{w}$ be vectors and k and ℓ be scalars. Then

1. closure under addition: $\underline{u}+\underline{v}$ is a vector
2. commutativity under addition: $\underline{u}+\underline{v} = \underline{v}+\underline{u}$
3. associativity of addition: $(\underline{u}+\underline{v})+\underline{w} = \underline{u}+(\underline{v}+\underline{w})$
4. zero: $\underline{u}+\underline{0} = \underline{0}+\underline{u} = \underline{u}$

⑤ negative: $\underline{u} + (-\underline{u}) = (-\underline{u}) + \underline{u} = \underline{0}$

⑥ closure under scalar multiplication: $k\underline{u}$ is a vector

⑦ scalar associativity: $k(l\underline{u}) = (kl)\underline{u}$

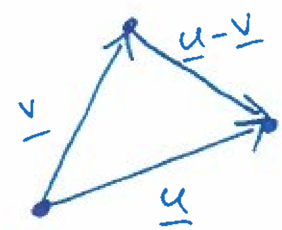
⑧ unity: $1\underline{u} = \underline{u}$

⑨ distributivity: $k(\underline{u}+\underline{v}) = k\underline{u} + k\underline{v}$

$$(k+l)\underline{u} = k\underline{u} + l\underline{u}$$

We can now define <u>vector subtraction</u>:

$$\underline{u} - \underline{v} = \underline{u} + (-\underline{v})$$



eg Simplify $5(\underline{u} - 2\underline{v}) + 6(5\underline{u} + 2\underline{v})$

$$= (5\underline{u} - 10\underline{v}) + (30\underline{u} + 12\underline{v})$$

$$\boxed{= 35\underline{u} + 2\underline{v}}$$

eg If $\underline{u} = 4\underline{v}$ show that $\underline{v} = \frac{1}{4}\underline{u}$.

We have $\frac{1}{4}\underline{u} = \frac{1}{4}(4\underline{v}) = (\frac{1}{4}\cdot 4)\underline{v} = 1\underline{v} = \underline{v}$.

Given vectors $\underline{u}$ and $\underline{v}$, a <u>linear combination</u> of them is a vector of the form $a\underline{u} + b\underline{v}$ where a and b are scalars.

More generally, a linear combination of the n vectors $\underline{u}_1, \underline{u}_2, \underline{u}_3, \ldots, \underline{u}_n$ is a vector of the form

$$k_1\underline{u}_1 + k_2\underline{u}_2 + k_3\underline{u}_3 + \cdots + k_n\underline{u}_n$$

where $k_1, k_2, k_3, \ldots, k_n$ are scalars.

eg Consider $\underline{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{u}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\underline{u}_3 = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

Choose scalars $k_1 = 4$, $k_2 = -1$, $k_3 = -1/2$.

Then
$$4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 3 \end{bmatrix} + \left(-\tfrac{1}{2}\right)\begin{bmatrix} 8 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

So $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is a linear combination of $\underline{u}_1$, $\underline{u}_2$ and $\underline{u}_3$.

eg Determine whether $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

We want to find scalars a, b for which
$$a\begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} 2a + 4b \\ a + 2b \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Thus $2a + 4b = 1$ and $a + 2b = 3$.

From the second equation, $a = 3 - 2b$

We substitute into the first equation: $2(3-2b) + 4b = 1$
$$6 - 4b + 4b = 1$$
$$6 = 1$$

There are no solutions, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is not a linear combination.

eg Determine whether $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

We want to find scalars a, b for which

$$a\begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2a+4b \\ a+3b \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Thus $2a+4b=1$ and $a+3b=3$

$$a = 3-3b$$

So $2(3-3b)+4b=1$

$$6-6b+4b=1$$

$$-2b=-5 \rightarrow b = 5/2$$

$$a = 3-3\left(\tfrac{5}{2}\right) = -\tfrac{9}{2}$$

Yes, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ <u>is</u> a linear combination of these vectors

because $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = -\frac{9}{2}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

<u>Theorem</u>: Given any two 2-dimensional vectors $\underline{u}$ and $\underline{v}$ which are not parallel, the set of linear combinations of $\underline{u}$ and $\underline{v}$ is the entire xy-plane.

Thus any two non-parallel vectors can be used as "building blocks" for any vector. In particular, we usually use the <u>standard basis vectors</u> $\underline{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\underline{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Here, $\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\underline{i} + b\underline{j}$.

In a similar way, we can consider 3-dimensional vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. The set of all linear combinations of two non-parallel 3-dimensional vectors is a plane which may or may not be the xy-plane. The set of all linear combinations of three non-parallel 3-dimensional vectors is <u>3-space</u>.

The standard basis vectors here are $\underline{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\underline{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ So $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\underline{i} + b\underline{j} + c\underline{k}$.

eg Determine whether $\begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$.

We set $a\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + b\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} a+3b \\ 2a+5b \\ -a+2b \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$$

Then $a+3b=1 \rightarrow a=1-3b$

$2a+5b=7 \rightarrow 2(1-3b)+5b=7$

$-a+2b=1 \qquad 2-6b+5b=7$

$-b=5 \rightarrow b=-5$

$a=1-3(-5)=16$

Now we must check that the 3rd equation is satisfied:

$$-a+2b=-16+2(-5)=-26 \neq 1$$

Hence there is no solution and $\begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$ is <u>not</u> a linear combination.