Please remember that each final exam is different, and the style and emphasis of the questions will vary from semester to semester.

1. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

[3] (a) Find a unit vector in the opposite direction to $\mathbf{u}$.

[4] (b) Find the exact angle between $\mathbf{u}$ and $\mathbf{v}$.

[4] (c) Find all k such that $\mathbf{u} + k\mathbf{v}$ is orthogonal to $\mathbf{u} - k\mathbf{v}$.

2. Consider the plane π which passes through the points $A(1, -1, 2)$, $B(-2, 2, 2)$, $C(-1, 0, 1)$.

[5] (a) Find the equation of the plane π.

[5] (b) Find an equation of the line ℓ which passes through the point $D(2, 3, 4)$ and which is orthogonal to π. Determine the point at which it intersects with π.

[6] (c) Find the distance from the point $P(3, 2, 1)$ to π, and find the point Q in the plane which is closest to P.

[5] 3. Find the matrix A, given $\left(A^T - 3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

[6] 4. Find conditions on the constant k such that the system

$$\left.\begin{array}{rcrcrcr} & & y & + & 2kz & = & 0 \\ x & + & 2y & + & 6z & = & 2 \\ kx & & & + & 2z & = & 1 \end{array}\right\}$$

has

(a) an infinite number of solutions

(b) no solutions

(c) a unique solution

[6] 5. (a) Consider the following system of linear equations:

$$\left.\begin{array}{rcrcrcrcrcr} x_1 & - & x_2 & & & + & 2x_4 & + & x_5 & = & 2 \\ -2x_1 & + & 2x_2 & + & x_3 & - & 4x_4 & & & = & -7 \\ x_1 & - & x_2 & + & x_3 & + & 3x_4 & + & x_5 & = & -1. \end{array}\right\}$$

Use Gaussian elimination to express the solution as a vector or as a linear combination of vectors.

[2] (b) Using your answer to part (a), determine the solution of the corresponding homogeneous system of equations:

$$\left.\begin{array}{rcrcrcrcrcl} x_1 & - & x_2 & & & + & 2x_4 & + & x_5 & = & 0 \\ -2x_1 & + & 2x_2 & + & x_3 & - & 4x_4 & & & = & 0 \\ x_1 & - & x_2 & + & x_3 & + & 3x_4 & + & x_5 & = & 0. \end{array}\right\}$$

[2] (c) Write $\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$, and $\mathbf{v}_5 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

[2] (d) Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$ linearly dependent or linearly independent? Explain.

[6] 6. (a) Use Gaussian elimination to find the inverse of the matrix $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$.

[4] (b) Using your results from part (a), solve the matrix equation

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

[5] 7. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$. Factor A as a product of elementary matrices.

[5] 8. (a) Find $\det(A)$, given $A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 2 & 0 & 0 & 4 \\ 3 & -1 & 2 & 0 \\ 0 & -4 & 3 & -2 \end{bmatrix}$.

[4] (b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} a+c & 2c \\ b+d & 2d \end{bmatrix}$. If $\det(A) = 2$, find $\det(A^2 B^T A^{-1})$.

9. Let $A = \begin{bmatrix} -2 & 1 \\ 4 & 1 \end{bmatrix}$.

[3] (a) Find the characteristic polynomial of A, and then determine the eigenvalues of A.

[4] (b) For each eigenvalue of A, finding a corresponding eigenvector.

[3] (c) Identify an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$, or explain why this is not possible.

[4] 10. Find the eigenvalues λ_1 and λ_2 of the matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$. Simplify the expressions $\dfrac{\lambda_1}{\lambda_2}$ and $\dfrac{\lambda_2}{\lambda_1}$.

[4] 11. (a) Let π be a plane spanned by vectors $\mathbf{e}$ and $\mathbf{f}$. Prove that if a vector $\mathbf{v}$ is orthogonal to both $\mathbf{e}$ and $\mathbf{f}$ then it is orthogonal to every vector in π.

[4] (b) If $E^2 = E$ and $A = I - 2E$, show that $A^{-1} = A$.

[4] (c) Let A be a square matrix with eigenvalue λ. Prove that if $A^2 = I$ then $\lambda = 1$ or $\lambda = -1$.