MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 6 | **MATH 2000** | FALL 2018

SOLUTIONS

[5] 1. (a) Let $f(x) = \dfrac{1}{x(x^2+1)}$. This is certainly positive and continuous for $x \geq 1$. Furthermore,

$$f'(x) = -\frac{3x^2+1}{x^2(x^2+1)^2},$$

so $f(x)$ is decreasing because $f'(x) < 0$ for all $x \geq 1$. Hence the requirements of the Integral Test are met.

Now, using a partial fraction decomposition,

$$\begin{aligned}
\int_1^\infty f(x)\,dx &= \lim_{T\to\infty} \int_1^T \frac{1}{x(x^2+1)}\,dx \\
&= \lim_{T\to\infty} \int_1^T \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx \\
&= \lim_{T\to\infty} \left[\ln|x| - \frac{1}{2}\ln|x^2+1|\right]_1^T \\
&= \lim_{T\to\infty} \left[\ln(T) - \frac{1}{2}\ln(T^2+1) - \ln(1) + \frac{1}{2}\ln(2)\right] \\
&= \lim_{T\to\infty} \ln\left(\frac{T}{\sqrt{T^2+1}}\right) + \frac{1}{2}\ln(2) \\
&= \ln(1) + \frac{1}{2}\ln(2) \\
&= \frac{1}{2}\ln(2).
\end{aligned}$$

Since the integral is convergent, the given series is <u>convergent</u> as well.

[5] (b) Let $f(x) = \dfrac{2x^2+1}{x(x^2+1)}$. This is certainly positive and continuous for $x \geq 1$. Furthermore,

$$f'(x) = -\frac{2x^4+x^2+1}{x^2(x^2+1)^2},$$

so $f(x)$ is decreasing because $f'(x) < 0$ for all $x \geq 1$. Hence the requirements of the Integral Test are met.

Now, using a partial fraction decomposition,

$$\begin{aligned}
\int_1^\infty f(x)\,dx &= \lim_{T\to\infty} \int_1^T \frac{2x^2+1}{x(x^2+1)}\,dx \\
&= \lim_{T\to\infty} \int_1^T \left(\frac{1}{x} + \frac{x}{x^2+1}\right) dx \\
&= \lim_{T\to\infty} \left[\ln|x| + \frac{1}{2}\ln|x^2+1|\right]_1^T \\
&= \lim_{T\to\infty} \left[\ln(T) + \frac{1}{2}\ln(T^2+1) - \ln(1) - \frac{1}{2}\ln(2)\right] \\
&= \lim_{T\to\infty} \ln\left(T\sqrt{T^2+1}\right) - \frac{1}{2}\ln(2) \\
&= \infty.
\end{aligned}$$

Because the integral is divergent, we conclude that the given series is also <u>divergent</u>.

[5] (c) Let $f(x) = \dfrac{\ln(x)}{x^2}$. Observe that $f(x)$ is positive and continuous for $x \geq 2$. Additionally,

$$f'(x) = \frac{1 - 2\ln(x)}{x^3}.$$

Note that $\ln(x) > 1$ for $x \geq 2$, so $f(x)$ is decreasing because $f'(x) < 0$ for all $x \geq 2$. Using integration by parts,

$$\begin{aligned}
\int_2^\infty f(x)\,dx &= \lim_{T\to\infty} \int_2^T \frac{\ln(x)}{x^2}\,dx \\
&= \lim_{T\to\infty} \left[-\frac{\ln(x)}{x} - \frac{1}{x}\right]_2^T \\
&= \lim_{T\to\infty} \left[-\frac{\ln(T)}{T} - \frac{1}{T} + \frac{\ln(2)}{2} + \frac{1}{2}\right] \\
&= \lim_{T\to\infty} \left[-\frac{\ln(T)}{T} - 0 + \frac{\ln(2)}{2} + \frac{1}{2}\right] \\
&\overset{\text{H}}{=} \frac{\ln(2)}{2} + \frac{1}{2} - \lim_{T\to\infty} \frac{\frac{1}{T}}{1} \\
&= \frac{\ln(2)+1}{2} - 0 \\
&= \frac{\ln(2)+1}{2}.
\end{aligned}$$

Because the integral is convergent, we know that the given series is <u>convergent</u> as well.

[5] 2. Let $f(x) = \dfrac{1}{\sqrt{x}e^{\sqrt{x}}}$, which is positive and continuous for $x \geq 1$. Observe that

$$f'(x) = -\frac{1+\sqrt{x}}{2x^{\frac{3}{2}}e^{\sqrt{x}}}$$

so $f(x)$ is decreasing since $f'(x) < 0$ for all $x \geq 1$. Hence the remainder estimate for the Integral Test applies. Thus we know that the nth remainder R_n is such that

$$R_n \leq \int_n^\infty f(x)\,dx,$$

where (by u-substitution with $u = -\sqrt{x}$)

$$\begin{aligned}
\int_n^\infty f(x)\,dx &= \lim_{T\to\infty} \int_n^T \frac{1}{\sqrt{x}e^{\sqrt{x}}}\,dx \\
&= \lim_{T\to\infty} -2\int_{-\sqrt{n}}^{-\sqrt{T}} e^u\,du \\
&= \lim_{T\to\infty} -2\Big[e^u\Big]_{-\sqrt{n}}^{-\sqrt{T}} \\
&= \lim_{T\to\infty} -2\left[e^{-\sqrt{T}} - e^{-\sqrt{n}}\right] \\
&= -2\left[0 - e^{-\sqrt{n}}\right] \\
&= \frac{2}{e^{\sqrt{n}}}.
\end{aligned}$$

When $n = 100$, then, we know that

$$R_{100} \leq \frac{2}{e^{\sqrt{100}}} = \frac{2}{e^{10}} \approx 0.0000908.$$

Hence the partial sum s_{100} is accurate to the true sum of the series with an error of no more than approximately 0.00009.

(In fact, the true sum of the series is about 0.94853967, while $s_{100} \approx 0.94845112$, so the true error is approximately 0.0000886, only slightly less than our "worst case scenario".)

[5] 3. (a) Observe that

$$\frac{1}{i(i^2+1)} = \frac{1}{i^3+i} \approx \frac{1}{i^3}$$

so we use the Direct Comparison Test with the test series $\sum \dfrac{1}{i^3}$ (a convergent p-series). Since $i^3 < i^3 + 1$, we immediately have

$$\frac{1}{i^3} > \frac{1}{i^3+1}$$

and thus we can conclude that the given series is convergent. (We could also use the Limit Comparison Test here.)

[5] (b) Observe that

$$\frac{2i^2+1}{i(i^2+1)} = \frac{2i^2+1}{i^3+i} \approx \frac{2i^2}{i^3} = \frac{2}{i}$$

so we use the Limit Comparison Test with the test series $\sum \frac{1}{i}$, the (divergent) harmonic series. Since

$$\lim_{i\to\infty} \frac{a_i}{t_i} = \lim_{i\to\infty} \frac{2i^2+1}{i(i^2+1)} \cdot i = \lim_{i\to\infty} \frac{2i^2+1}{i^2+1} = 2,$$

we can conclude that the given series is also <u>divergent</u>.

[5] (c) Observe that

$$i! = 1\cdot 2\cdot 3\cdots i > 1\cdot \underbrace{2\cdot 2\cdots 2}_{(i-1)\text{ times}} = 2^{i-1}$$

so

$$\frac{1}{i!} < \frac{1}{2^{i-1}} = \left(\frac{1}{2}\right)^{i-1}.$$

Thus we employ the Direct Comparison Test with the test series $\sum \left(\frac{1}{2}\right)^{i-1}$ (a convergent geometric series) to conclude that the given series is also <u>convergent</u>.

[5] (d) First we note that this series consists only of negative terms, so we write

$$\sum_{i=3}^{\infty} \frac{2^{i-1}(4i^2-5)}{6^{i+1}(2i-i^2)} = -\sum_{i=3}^{\infty} \frac{2^{i-1}(4i^2-5)}{6^{i+1}(i^2-2i)}$$

and we will instead apply a Comparison Test to the resulting positive series. For large i,

$$\frac{4i^2-5}{i^2-2i} \approx 4,$$

while

$$\frac{2^{i-1}}{6^{i+1}} \approx \left(\frac{2}{6}\right)^i = \left(\frac{1}{3}\right)^i.$$

Since the geometric terms dominate, we will use the Limit Comparison Test and choose as our test series $\sum \left(\frac{1}{3}\right)^i$ (a convergent geometric series). Thus

$$\lim_{i\to\infty} \frac{a_i}{t_i} = \lim_{i\to\infty} \frac{2^{i-1}(4i^2-5)}{6^{i+1}(i^2-2i)} \cdot 3^i = \lim_{i\to\infty} \frac{2^{-1}\cdot 2^i \cdot 3^i}{6\cdot 6^i} \cdot \lim_{i\to\infty} \frac{4i^2-5}{i^2-2i} = \frac{1}{12}\cdot 4 = \frac{1}{3}.$$

Now we can conclude that the given series is also <u>convergent</u>.