# MEMORIAL UNIVERSITY OF NEWFOUNDLAND

## DEPARTMENT OF MATHEMATICS AND STATISTICS

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Assignment 5 **MATH 2000** Fall 2018

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## SOLUTIONS

[6] 1. (a) We can write

$$\begin{aligned}\sum_{i=0}^{\infty}(-1)^{i-1}\frac{2^{3i+4}-3^i}{9^{i-1}} &= \sum_{i=0}^{\infty}(-1)^{-1}\cdot(-1)^i\frac{2^4\cdot 8^i-3^i}{9^{-1}\cdot 9^i} \\ &= -\sum_{i=0}^{\infty}\frac{16(-8)^i-(-3)^i}{\frac{1}{9}\cdot 9^i} \\ &= -144\sum_{i=0}^{\infty}\left(-\frac{8}{9}\right)^i+9\sum_{i=0}^{\infty}\left(-\frac{1}{3}\right)^i.\end{aligned}$$

Now both the series on the righthand side are convergent geometric series, since their common ratios — respectively, $-\frac{8}{9}$ and $-\frac{1}{3}$ — are such that $|r|<1$. Thus

$$\begin{aligned}\sum_{i=0}^{\infty}(-1)^{i-1}\frac{2^{3i+4}-3^i}{9^{i-1}} &= -144\cdot\frac{1}{1+\frac{8}{9}}+9\cdot\frac{1}{1+\frac{1}{3}} \\ &= -144\cdot\frac{9}{17}+9\cdot\frac{3}{4} \\ &= -\frac{4725}{68}.\end{aligned}$$

[2] (b) This series is identical to the series in part (a), except that it is missing the terms corresponding to $i=0$ and $i=1$. When $i=0$, the term is $a_0=-135$. When $i=1$, the term is $a_1=125$. Thus

$$\sum_{i=2}^{\infty}(-1)^{i-1}\frac{2^{3i+4}-3^i}{9^{i-1}} = -\frac{4725}{68}-(-135)-125=-\frac{4045}{68}.$$

[3] (c) We would first like to evaluate $\lim\limits_{i\to\infty} a_i$. The corresponding function is

$$f(x)=\frac{2^x}{x^2},$$

so

$$\lim_{x\to\infty} f(x) \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2^x\ln(2)}{2x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2^x\ln^2(2)}{2}=\infty.$$

Although the Evaluation Theorem does not directly apply, this tells us that the values of the function are becoming unboundedly large, which means that the terms of the sequence (being points on the graph of this function) must become unboundedly large as well. Thus $\lim\limits_{i\to\infty} a_i$ does not exist, and so the given series is <u>divergent</u> by the Divergence Test.

[5] (d) We can write

$$\sum_{i=1}^{\infty} \frac{4 \cdot 8 \cdot 12 \cdots (4i)}{6 \cdot 12 \cdot 18 \cdots (6i)} = \sum_{i=1}^{\infty} \frac{4^i i!}{6^i i!} = \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^i = \frac{2}{3} \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1}.$$

This is a convergent geometric series with common ratio $\frac{2}{3}$, so

$$\sum_{i=1}^{\infty} \frac{4 \cdot 8 \cdot 12 \cdots (4i)}{6 \cdot 12 \cdot 18 \cdots (6i)} = \frac{2}{3} \cdot \frac{1}{1 - \frac{2}{3}} = \frac{2}{3} \cdot 3 = 2.$$

[6] 2. We can write

$$\sum_{i=0}^{\infty} \frac{(4x)^{3i}}{8^i} = \sum_{i=0}^{\infty} \left(\frac{64x^3}{8}\right)^i = \sum_{i=0}^{\infty} (8x^3)^i.$$

This is a geometric series, so it will converge only if

$$\begin{array}{rcccl} -1 & < & 8x^3 & < & 1 \\ -\frac{1}{8} & < & x^3 & < & \frac{1}{8} \\ -\frac{1}{2} & < & x & < & \frac{1}{2}. \end{array}$$

For these values of $x$, the sum of the series will be

$$\sum_{i=0}^{\infty} \frac{(4x)^{3i}}{8^i} = \frac{1}{1 - 8x^3}.$$

[6] 3. We can write

$$\begin{aligned} 2.018018018\ldots &= 2 + 0.018 + 0.000018 + 0.000000018 + \cdots \\ &= 2 + 0.018(1 + 0.001 + 0.000001 + \cdots) \\ &= 2 + \frac{18}{1000} \sum_{i=1}^{\infty} \left(\frac{1}{1000}\right)^{i-1} \\ &= 2 + \frac{18}{1000} \cdot \frac{1}{1 - \frac{1}{1000}} \\ &= 2 + \frac{18}{1000} \cdot \frac{1000}{999} \\ &= \frac{224}{111}. \end{aligned}$$

[5] 4. (a) First we need to find the critical points. We have

$$f_x(x, y) = 2 + 2x \quad \text{and} \quad f_y(x, y) = -4 + 8y.$$

Setting these equal to zero gives $x = -1$ and $y = \frac{1}{2}$, so the only critical point is $(-1, \frac{1}{2})$.
Next,

$$f_{xx}(x, y) = 2, \quad f_{yy}(x, y) = 8, \quad \text{and} \quad f_{xy}(x, y) = 0.$$

Hence the discriminant at $(-1, \frac{1}{2})$ is $D = 16 > 0$ and $f_{xx}(-3, 2) = 2 > 0$. This means that $(-1, \frac{1}{2})$ is a relative minimum.

[7] (b) We have

$$f_x(x, y) = 16y - 2y^2 - 16xy + 2xy^2 \quad \text{and} \quad f_y(x, y) = 16x - 4xy - 8x^2 + 2x^2y.$$

We can factor the first equation as

$$-2y(y - 8) + 2xy(y - 8) = 2y(y - 8)(x - 1),$$

so this equation is equal to zero only for $y = 0$, $y = 8$ or $x = 1$. Substituting each of these into the second equation and setting it equal to zero, we find that the critical points are $(0, 0)$, $(2, 0)$, $(0, 8)$, $(2, 8)$ and $(1, 4)$.
Next,

$$f_{xx}(x, y) = -16y + 2y^2, \quad f_{yy}(x, y) = -4x + 2x^2, \quad f_{xy}(x, y) = 16 - 4y - 16x + 4xy.$$

At the first four critical points, the discriminant is $D = -256 < 0$, so these are all saddle points. However, at $(1, 4)$, $D = 64 > 0$ and $f_{xx}(1, 4) = -32 < 0$ so $(1, 4)$ is a relative maximum.