MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2 | **MATH 2000** | FALL 2018

SOLUTIONS

[4] 1. (a) We compare the given series to $\sum \frac{4^i}{5^i} = \sum \left(\frac{4}{5}\right)^i$, which is a convergent geometric series. Then because $\sqrt{i} \geq 0$,

$$\frac{4^i}{\sqrt{i}+5^i} < \frac{4^i}{5^i}$$

and so the given series <u>converges</u> by the Direct Comparison Test.

[4] (b) We compare the given series to $\sum \frac{2i}{4i^2} \approx \sum \frac{1}{i}$, which is a divergent p-series. (We could also include the coefficient of $\frac{1}{2}$, but this is not necessary.) Then

$$\lim_{i\to\infty} \frac{a_i}{t_i} = \lim_{i\to\infty} \frac{2i+7}{4i^2+3i} \cdot i = \lim_{i\to\infty} \frac{2i^2+7i}{4i^2+3i} = \frac{1}{2},$$

so the given series <u>diverges</u> by the Limit Comparison Test.

[6] (c) Consider the function

$$f(x) = \frac{\ln(x)+1}{x^4}.$$

This is continuous and positive for $x \geq 1$. Furthermore,

$$f'(x) = \frac{\frac{1}{x}\cdot x^4 - 4x^3[\ln(x)+1]}{x^8} = \frac{-4\ln(x)-3}{x^5} = -\frac{4\ln(x)+3}{x^5}.$$

Thus $f'(x) < 0$ and so $f(x)$ is decreasing for $x \geq 1$. Hence the given series meets the requirements of the Integral Test. Then

$$\int_1^\infty f(x) = \lim_{T\to\infty} \int_1^T \frac{\ln(x)+1}{x^4}\, dx.$$

We use integration by parts with $w = \ln(x)+1$ so $dw = \frac{1}{x}\,dx$ and $dv = \frac{1}{x^4}\,dx$ so

$v = -\dfrac{1}{3x^3}$. Thus

$$\begin{aligned}
\int_1^\infty f(x)\,dx &= \lim_{T\to\infty}\left(\left[-\frac{1}{3x^3}\cdot[\ln(x)+1]\right]_1^T - \int_1^T -\frac{1}{3x^3}\cdot\frac{1}{x}\,dx\right)\\
&= \lim_{T\to\infty}\left(\left[-\frac{1}{3x^3}\cdot[\ln(x)+1]\right]_1^T + \frac{1}{3}\int_1^T \frac{1}{x^4}\,dx\right)\\
&= \lim_{T\to\infty}\left[-\frac{\ln(x)}{3x^3} - \frac{1}{3x^3} - \frac{1}{9x^3}\right]_1^T\\
&= \lim_{T\to\infty}\left[-\frac{\ln(x)}{3x^3} - \frac{4}{9x^3}\right]_1^T\\
&= \lim_{T\to\infty}\left[-\frac{\ln(T)}{3T^3} - \frac{4}{9T^3} + 0 + \frac{4}{9}\right]\\
&= \frac{4}{9} - \frac{1}{3}\lim_{T\to\infty}\frac{\ln(T)}{T^3}\\
&\overset{\text{H}}{=} \frac{4}{9} - \frac{1}{3}\lim_{T\to\infty}\frac{\frac{1}{T}}{3T^2}\\
&= \frac{4}{9} - \frac{1}{9}\lim_{T\to\infty}\frac{1}{T^3}\\
&= \frac{4}{9}.
\end{aligned}$$

Since the improper integral converges, the given series also <u>converges</u> by the Integral Test.

Alternatively, a less obvious way to approach this problem is to observe that $\ln(i)$ increases as $i \to \infty$, so perhaps we can assume $\ln(i) + 1 \approx i$ and compare the given series to $\sum \dfrac{i}{i^4} = \sum \dfrac{1}{i^3}$, a convergent p-series. Then

$$\lim_{i\to\infty}\frac{a_i}{t_i} = \lim_{i\to\infty}\frac{\ln(i)+1}{i^4}\cdot i^3 = \lim_{i\to\infty}\frac{\ln(i)+1}{i}.$$

Now we need l'Hôpital's Rule, so we consider the limit of the corresponding function:

$$\lim_{x\to\infty}\frac{\ln(x)+1}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x} = 0$$

so

$$\lim_{i\to\infty}\frac{a_i}{t_i} = 0$$

by the Evaluation Theorem. Thus the given series again <u>converges</u> by the Limit Comparison Test.

[4] 2. (a) We can rewrite the given series as

$$\sum_{i=0}^{\infty} \frac{(-1)^{i+1}6^i}{9^{i-1}} = \sum_{i=0}^{\infty} \frac{(-1)^i \cdot (-1) \cdot 6^i}{9^i \cdot 9^{-1}} = -9 \sum_{i=0}^{\infty} \left(-\frac{6}{9}\right)^i = -9 \sum_{i=0}^{\infty} \left(-\frac{2}{3}\right)^i.$$

This is a convergent geometric series with sum

$$-9 \cdot \frac{1}{1-\left(-\frac{2}{3}\right)} = -9 \cdot \frac{3}{5} = -\frac{27}{5}.$$

[6] (b) Observe that

$$\frac{2}{4i^2+8i+3} = \frac{2}{(2i+1)(2i+3)} = \frac{A}{2i+1} + \frac{B}{2i+3}.$$

Thus

$$2 = A(2i+3) + B(2i+1).$$

When $i = -\frac{1}{2}$, we have $2 = A(2)$ so $A = 1$. When $i = -\frac{3}{2}$, we have $2 = B(-2)$ so $B = -1$. Thus

$$\sum_{i=1}^{\infty} \frac{2}{4i^2+8i+3} = \sum_{i=1}^{\infty} \left(\frac{1}{2i+1} - \frac{1}{2i+3}\right).$$

Then

$$\begin{aligned}
s_1 &= \frac{1}{3} - \frac{1}{5} \\
s_2 &= \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) = \frac{1}{3} - \frac{1}{7} \\
s_3 &= \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) = \frac{1}{3} - \frac{1}{9}
\end{aligned}$$

and, in general,

$$s_n = \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) = \frac{1}{3} - \frac{1}{2n+3}.$$

Thus

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{3} - \frac{1}{2n+3}\right) = \frac{1}{3}$$

and so the telescoping series must converge with sum $\frac{1}{3}$.

[1] 3. (a) We are told that the series is convergent so, by the contrapositive of the Divergence Test,

$$\lim_{i\to\infty} a_i = 0.$$

[3] (b) Since this is a positive series, we can compare it with $\sum a_i$, which we are told converges. Then we have

$$\lim_{i\to\infty} \frac{(a_i)^2}{a_i} = \lim_{i\to\infty} a_i = 0$$

by part (a). Hence, by the Limit Comparison Test, $\sum (a_i)^2$ also <u>converges</u>.

[4] 4. Observe that x and z both depend on q while y does not. Thus, by the Chain Rule,

$$\frac{\partial w}{\partial q} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial q} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial q}.$$

Here

$$\begin{aligned} \frac{\partial w}{\partial x} &= \ln(y) \\ \frac{\partial w}{\partial z} &= 3z^2 \\ \frac{\partial x}{\partial q} &= p\cos(q) \\ \frac{\partial z}{\partial q} &= -7 \end{aligned}$$

so

$$\begin{aligned} \frac{\partial w}{\partial q} &= \ln(y) \cdot p\cos(q) + 3z^2 \cdot (-7) \\ &= p\ln(y)\cos(q) - 21z^2. \end{aligned}$$

[8] 5. We have

$$\begin{aligned} f_x(x,y) &= \frac{1}{9}x^2 + \frac{1}{9}y - \frac{10}{3} \\ f_y(x,y) &= \frac{1}{9}x - \frac{1}{9}y. \end{aligned}$$

Setting $f_x(x,y) = 0$ and $f_y(x,y) = 0$, the second equation yields

$$\frac{1}{9}x - \frac{1}{9}y = 0 \quad \Longrightarrow \quad y = x.$$

Substituting this into the first equation, we have

$$\begin{aligned} \frac{1}{9}x^2 + \frac{1}{9}y - \frac{10}{3} &= 0 \\ \frac{1}{9}x^2 + \frac{1}{9}x - \frac{10}{3} &= 0 \\ x^2 + x - 30 &= 0 \\ (x+6)(x-5) &= 0 \end{aligned}$$

so either $x = -6$ or $x = 5$. Thus the critical points are $(-6,-6)$ and $(5,5)$.

Next we have

$$\begin{aligned} f_{xx}(x,y) &= \frac{2}{9}x \\ f_{xy}(x,y) &= \frac{1}{9} \\ f_{yy}(x,y) &= -\frac{1}{9}. \end{aligned}$$

For $(x,y) = (-6,-6)$ we have

$$D = \left(-\frac{4}{3}\right) \cdot \left(-\frac{1}{9}\right) - \left(\frac{1}{9}\right)^2 = \frac{11}{81} > 0,$$

so this is a <u>local maximum</u> because $f_{xx}(-6,-6) = -\frac{4}{3} < 0$.
For $(x,y) = (5,5)$ we have

$$D = \left(\frac{10}{9}\right) \cdot \left(-\frac{1}{9}\right) - \left(\frac{1}{9}\right)^2 = -\frac{11}{81} < 0,$$

so this is a <u>saddle point</u>.