MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 2.5 | **Math 2000 Worksheet** | FALL 2018

SOLUTIONS

1. (a) First we find the first-order partial derivatives:

$$f_x(x,y) = 6x^5 - 6y \quad \text{and} \quad f_y(x,y) = 6y^5 - 6x.$$

Setting the first of these equal to zero gives $6x^5 - 6y = 0$ so $y = x^5$ and so we obtain from the second equation $6(x^5)^5 - 6x = 6x^{25} - 6x = 6x(x^{24} - 1) = 0$ so $x = 0$ (and $y = 0$), $x = 1$ (and $y = 1$), or $x = -1$ (and $y = -1$), giving us three critical points: $(0,0)$, $(1,1)$ and $(-1,-1)$.

To determine the nature of each of these critical points, we require the second derivatives:

$$f_{xx}(x,y) = 30x^4, \quad f_{yy}(x,y) = 30y^4, \quad f_{xy}(x,y) = -6.$$

For $(0,0)$ this gives

$$D = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 0 - 36 = -36 < 0,$$

so $(x,y) = (0,0)$ is a saddle point. For $(1,1)$ we have

$$D = f_{xx}(1,1)f_{yy}(1,1) - [f_{xy}(1,1)]^2 = 900 - 36 = 864 > 0$$

and since $f_{xx}(1,1) = 30 > 0$, $(x,y) = (1,1)$ is a local minimum. For $(-1,-1)$, we also get

$$D = f_{xx}(-1,-1)f_{yy}(-1,-1) - [f_{xy}(-1,-1)]^2 = 900 - 36 = 864 > 0$$

with $f_{xx}(-1,-1) = 30 > 0$, so $(x,y) = (-1,-1)$ is also a local minimum.

(b) We again begin by computing the first-order partial derivatives:

$$f_x(x,y) = 3x^2y^2 - 36x \quad \text{and} \quad f_y(x,y) = 2x^3y - 54y.$$

Setting the latter equal to zero gives $2y(x^3 - 27) = 0$ and hence either $y = 0$ or $x = 3$. If $y = 0$, setting $f_x(x,y) = 0$ implies $-36x = 0$ so $x = 0$. By the same process, if $x = 3$, we get $27y^2 - 108 = 0$ so $y^2 = 4$ and either $y = 2$ or $y = -2$. Thus the three critical points are $(0,0)$, $(3,2)$ and $(3,-2)$.

The second derivatives are

$$f_{xx}(x,y) = 6xy^2 - 36, \quad f_{yy}(x,y) = 2x^3 - 54, \quad f_{xy}(x,y) = 6x^2y.$$

For $(0,0)$ we have

$$D = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = (-36)(-54) - 0 = 1944 > 0$$

and since $f_{xx}(x,y) = -36 < 0$, $(x,y) = (0,0)$ is a local maximum. For $(3,2)$ we have

$$D = f_{xx}(3,2)f_{yy}(3,2) - [f_{xy}(3,2)]^2 = (36)(0) - (108)^2 = -11664 < 0,$$

so $(x,y) = (3,2)$ is a saddle point. For $(3,-2)$, we also get

$$D = f_{xx}(3,-2)f_{yy}(3,-2) - [f_{xy}(3,-2)]^2 = (36)(0) - (108)^2 = -11664 < 0,$$

so $(x,y) = (3,-2)$ is another saddle point.