# MEMORIAL UNIVERSITY OF NEWFOUNDLAND

## DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 1 | **MATH 2000** | FALL 2018

## SOLUTIONS

[4] 1. (a) We have

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} \frac{5\cdot 2^{3i+1}}{7^i - 3^{2i-1}} = \lim_{i\to\infty} \frac{5\cdot 2^{3i}\cdot 2}{7^i - 3^{2i}\cdot 3^{-1}} = \lim_{i\to\infty} \frac{10\cdot 8^i}{7^i - \frac{1}{3}\cdot 9^i}.$$

Since $9^i$ is the dominant term in the denominator, we write

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} \frac{10\cdot 8^i}{7^i - \frac{1}{3}\cdot 9^i}\cdot\frac{\frac{1}{9^i}}{\frac{1}{9^i}} = \lim_{i\to\infty} \frac{10\left(\frac{8}{9}\right)^i}{\left(\frac{7}{9}\right)^i - \frac{1}{3}} = \frac{10\cdot 0}{0-\frac{1}{3}} = 0.$$

[5] (b) Observe that

$$\lim_{i\to\infty} |a_i| = \lim_{i\to\infty} \frac{(i-3)^2}{i(i^2+4)} = \lim_{i\to\infty} \frac{i^2-6i+9}{i^3+4i}\cdot\frac{\frac{1}{i^3}}{\frac{1}{i^3}} = \lim_{i\to\infty} \frac{\frac{1}{i}-\frac{6}{i^2}+\frac{9}{i^3}}{1+\frac{4}{i^2}} = \frac{0-0+0}{1+0} = 0.$$

Thus, by the Absolute Sequence Theorem,

$$\lim_{i\to\infty} a_i = 0$$

as well.

[5] (c) The corresponding function is

$$f(x) = \frac{\ln(e^x+2)}{e^x+3}.$$

By l'Hôpital's Rule,

$$\lim_{x\to\infty} f(x) \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{e^x+2}\cdot e^x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x+2} = 0.$$

Since this limit exists, by the Evaluation Theorem we can conclude that

$$\lim_{i\to\infty} a_i = \lim_{x\to\infty} f(x) = 0.$$

[6] 2. The corresponding function is

$$f(x) = 3 - \frac{8x}{e^{2x}}$$

so

$$f'(x) = 0 - \frac{8e^{2x} - 2e^{2x}(8x)}{(e^{2x})^2} = \frac{16xe^{2x} - 8e^{2x}}{e^{4x}} = \frac{8(2x-1)}{e^{2x}}.$$

Since the denominator is always positive and $2x - 1 > 0$ for $x \geq 1$, $f'(x) > 0$ for $x \geq 1$ and therefore $f(x)$ is increasing for $x \geq 1$. Hence $\{a_i\}$ must also be monotonic increasing.

Immediately, then, we know that $a_1 = 3 - \frac{8}{e^2}$ is a lower bound. Furthermore,

$$\frac{8i}{e^{2i}} > 0 \quad\Longrightarrow\quad a_i = 3 - \frac{8i}{e^{2i}} < 3 - 0 = 3$$

for all $i$. Hence $3 - \frac{8}{e^2} < a_i < 3$ and therefore $\{a_i\}$ is bounded.

By the Bounded Monotonic Sequence Theorem, then, $\{a_i\}$ must converge.

[6] 3. (a) There are several ways in which a non-monotonic sequence could converge. It could experience diminished oscillations around its limit; it could have a monotonic tail; or it could simply be a constant sequence. Figure 1 offers examples of each of these.

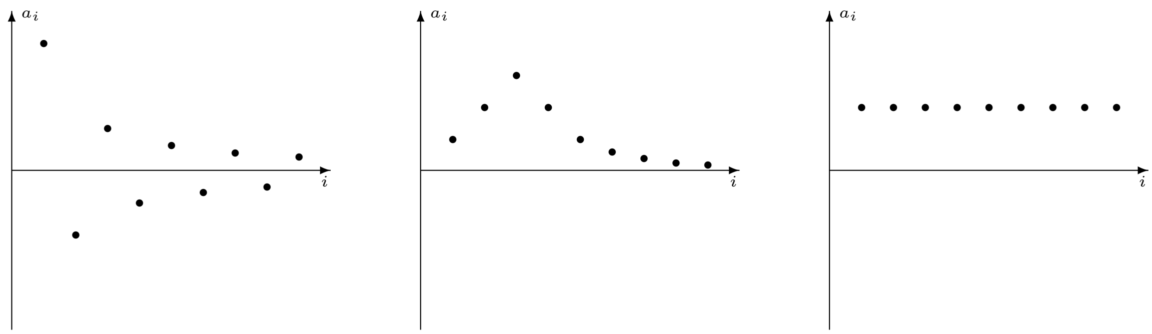
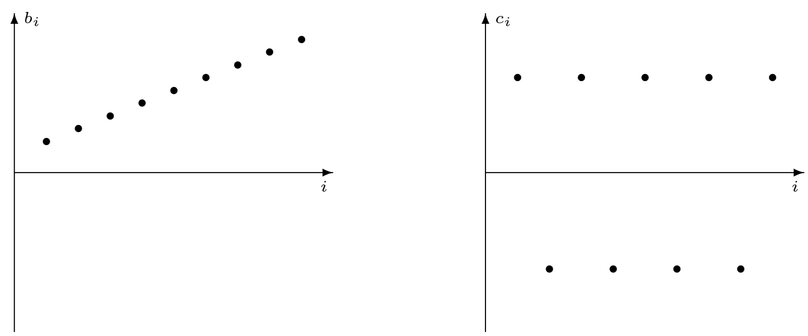



Figure 1: Three convergent, non-monotonic sequences as suggested by Question 3(a).

(b) Such a sequence could simply increase (or decrease) without bound, as in Figure 2(a).




(a) Suggested by Question 3(b)

(b) Suggested by Question 3(c)

Figure 2: Questions 3(b) and (c).

(c) Such a sequence could oscillate forever between the same two (or more) numbers, as in Figure 2(b).

[4] 4. In order for the numerator to be defined, we must have $y > 0$. In order for the denominator to be defined, we must have $x^2 - y \geq 0$, but to avoid division by zero we must restrict this to $x^2 - y > 0$ or $y < x^2$. Thus the domain of $f(x, y)$ consists of all points below the parabola $y = x^2$ (excluding the parabola itself) and above the $x$-axis. This is depicted in Figure 3.

[5] 5. Along the line $y = 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{6x^2y}{x^3 + 2y^3} = \lim_{x\to 0} \frac{0}{x^3 + 0} = \lim_{x\to 0} 0 = 0.$$

Along the line $x = 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{6x^2y}{x^3 + 2y^3} = \lim_{y\to 0} \frac{0}{0 + 2y^3} = \lim_{y\to 0} 0 = 0.$$

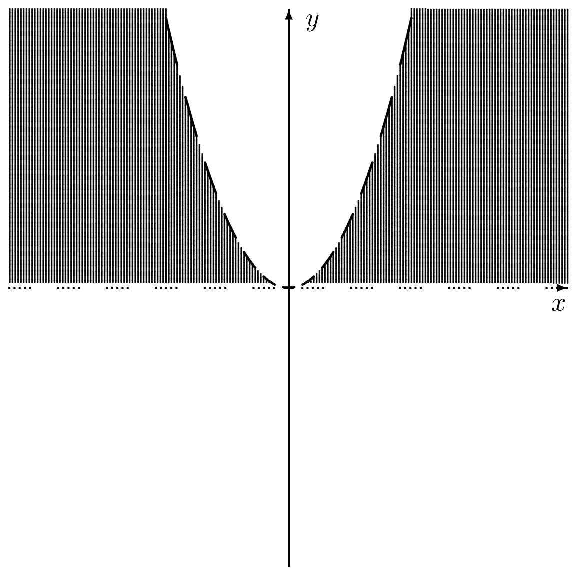



Figure 3: The domain of $f(x, y)$ for Question 4.

But along the line $y = x$, we have

$$\lim_{(x,y)\to(0,0)} \frac{6x^2y}{x^3+2y^3} = \lim_{x\to 0} \frac{6x^2 \cdot x}{x^3+2x^3} = \lim_{x\to 0} \frac{6x^3}{3x^3} = \lim_{x\to 0} 2 = 2.$$

Since there are at least two paths along which the limit is not equal, we can conclude that $\lim_{(x,y)\to(0,0)} \frac{6x^2y}{x^3+2y^3}$ does not exist.

[5] 6. We have

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{1}{x+y^2} \cdot 1 = \frac{1}{x+y^2} \\
\frac{\partial z}{\partial y} &= \frac{1}{x+y^2} \cdot 2y = \frac{2y}{x+y^2} \\
\frac{\partial^2 z}{\partial x^2} &= \frac{0(x+y^2) - 1(1)}{(x+y^2)^2} = -\frac{1}{(x+y^2)^2} \\
\frac{\partial^2 z}{\partial y^2} &= \frac{2(x+y^2) - 2y(2y)}{(x+y^2)^2} = \frac{2x-2y^2}{(x+y^2)^2} \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{0(x+y^2) - 2y(1)}{(x+y^2)^2} = -\frac{2y}{(x+y^2)^2}.
\end{aligned}$$

Thus

$$2x\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 2x\left(-\frac{1}{(x+y^2)^2}\right) + \frac{2x-2y^2}{(x+y^2)^2} = -\frac{2y^2}{(x+y^2)^2}$$

and

$$y\frac{\partial^2 z}{\partial x \partial y} = y\left(-\frac{2y}{(x+y^2)^2}\right) = -\frac{2y^2}{(x+y^2)^2}.$$

Since these are equal, $z = \ln(x + y^2)$ satisfies the PDE and we conclude that it <u>is a solution</u>.