MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SOLUTIONS

1. (a) We use Limit Comparison with the convergent geometric series $\displaystyle\sum_{i=0}^{\infty}\left(\frac{3}{7}\right)^i = \sum_{i=0}^{\infty}\frac{3^i}{7^i}$:

$$\lim_{i\to\infty}\frac{\frac{3^i}{3+7^i}}{\frac{3^i}{7^i}} = \lim_{i\to\infty}\frac{1}{3\left(\frac{1}{7}\right)^i+1} = 1 > 0$$

so the given series converges.

(b) Note that

$$\lim_{i\to\infty}\frac{i^{\frac{1}{i}}}{\arctan(i)} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \neq 0.$$

By the Divergence Test, the given series diverges.

(c) We use the Ratio Test, letting

$$a_i = \frac{i!}{5\cdot 9\cdot 13\cdots(4i+1)} \quad \text{so} \quad a_{i+1} = \frac{(i+1)!}{5\cdot 9\cdot 13\cdots(4i+1)\cdot(4i+5)}.$$

Then

$$\begin{aligned}\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| &= \lim_{i\to\infty}\frac{(i+1)!}{5\cdot 9\cdot 13\cdots(4i+1)\cdot(4i+5)}\cdot\frac{5\cdot 9\cdot 13\cdots(4i+1)}{i!}\\ &= \lim_{i\to\infty}\frac{i+1}{4i+5} = \frac{1}{4} = L.\end{aligned}$$

Since $L < 1$, the series converges.

(d) We use the Alternating Series Test. Let $a_i = \frac{1}{i\ln(i)}$. Clearly, $\lim\limits_{i\to\infty} a_i = 0$. To see that $\{a_i\}$ is decreasing, let $f(x) = \frac{1}{x\ln(x)}$ so then

$$f'(x) = -\frac{\ln(x)+1}{x^2\ln^2(x)} < 0 \quad \text{for all } x \geq 2.$$

Hence the given series converges.

(e) We use the Direct Comparison Test with the (divergent) harmonic series $\displaystyle\sum_{i=1}^{\infty}\frac{1}{i}$. Note that

$$\begin{aligned}\cos^2(i) &\geq 0\\ i\cos^2(i) &\geq 0\\ -i\cos^2(i) &\leq 0\\ i - i\cos^2(i) &\leq i\\ \frac{1}{i-i\cos^2(i)} &\geq \frac{1}{i}.\end{aligned}$$

So the given series diverges as well.

(f) We use the Root Test, letting $a_i = \left(\frac{2}{e^{-8i} - 1}\right)^i$. Then

$$\lim_{i\to\infty} |a_i|^{\frac{1}{i}} = \lim_{i\to\infty} \left|\frac{2}{e^{-8i} - 1}\right| = 2 = L,$$

so since $L > 1$ the given series diverges.