MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 1.3 **Math 2000 Worksheet** FALL 2018

SOLUTIONS

1. (a) We have

$$
\begin{aligned}
s_1 &= a_1 = -1 \\
s_2 &= a_1 + a_2 = -1 + \frac{3}{2} = \frac{1}{2} \\
s_3 &= a_1 + a_2 + a_3 = -1 + \frac{3}{2} + \frac{5}{7} = \frac{17}{14} \\
s_4 &= a_1 + a_2 + a_3 + a_4 = -1 + \frac{3}{2} + \frac{5}{7} + \frac{1}{2} = \frac{12}{7} \\
s_5 &= a_1 + a_2 + a_3 + a_4 + a_5 = -1 + \frac{3}{2} + \frac{5}{7} + \frac{1}{2} + \frac{9}{23} = \frac{339}{161}
\end{aligned}
$$

(b) Note that this series has $i = 3$ as its first index, so

$$
\begin{aligned}
s_1 &= a_3 = -\frac{1}{6} \\
s_2 &= a_3 + a_4 = -\frac{1}{6} + \frac{1}{24} = -\frac{1}{8} \\
s_3 &= a_3 + a_4 + a_5 = -\frac{1}{6} + \frac{1}{24} - \frac{1}{120} = -\frac{2}{15} \\
s_4 &= a_3 + a_4 + a_5 + a_6 = -\frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = -\frac{19}{144} \\
s_5 &= a_3 + a_4 + a_5 + a_6 + a_7 = -\frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} = -\frac{37}{280}
\end{aligned}
$$

2. (a) We have

$$\lim_{i\to\infty} \left(-\frac{3}{7}\right)^{i+1} = \lim_{i\to\infty} -\frac{3}{7}\left(-\frac{3}{7}\right)^{i} = 0,$$

so the Divergence Test yields no conclusion.

(b) We have

$$\lim_{i\to\infty} \frac{i^3}{i(4i^2-5)} = \lim_{i\to\infty} \frac{i^3}{4i^3-5i} = \frac{1}{4},$$

so by the Divergence Test, this series is divergent.

(c) We have

$$\lim_{k\to\infty} \frac{2^k}{5^{k-1}} = 5\lim_{k\to\infty} \left(\frac{2}{5}\right)^k = 0,$$

so we can draw no conclusion from the Divergence Test.

(d) We have

$$\lim_{i\to\infty} i\sin\left(\frac{1}{i}\right) = \lim_{i\to\infty} \frac{\sin\left(\frac{1}{i}\right)}{\frac{1}{i}} = 1,$$

so by the Divergence Test, this series is divergent.

3. First note that $a_1 = s_1 = 3$. For $n > 1$, we simply have

$$a_n = s_n - s_{n-1} = \left[5 - \frac{2}{n^2}\right] - \left[5 - \frac{2}{(n-1)^2}\right] = \frac{4n-2}{n^2(n-1)^2}.$$

Finally,

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[5 - \frac{2}{n^2}\right] = 5.$$

4. (a) Decomposing into partial fractions gives

$$\frac{1}{(i+4)(i+5)} = \frac{1}{i+4} - \frac{1}{i+5}.$$

Then we observe that this is a telescoping series with

$$s_n = \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n+3} - \frac{1}{n+4}\right) + \left(\frac{1}{n+4} - \frac{1}{n+5}\right) = \frac{1}{4} - \frac{1}{n+5}.$$

Hence

$$\sum_{i=0}^{\infty} \frac{1}{(i+4)(i+5)} = \lim_{n\to\infty}\left(\frac{1}{4} - \frac{1}{n+5}\right) = \frac{1}{4}.$$

(b) Again, this is a telescoping series, with

$$s_n = [1^2 - 2^2] + [2^2 - 3^2] + \cdots + [(n-1)^2 - n^2] + [n^2 - (n+1)^2] = 1 - (n+1)^2.$$

Hence

$$\sum_{n=1}^{\infty}\left[n^2 - (n+1)^2\right] = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[1 - (n+1)^2\right] = -\infty.$$

Therefore, the series diverges.

(c) Decomposing into partial fractions gives

$$\frac{2}{i(i-1)(i+1)} = \frac{1}{i-1} - \frac{2}{i} + \frac{1}{i+1}.$$

Then we observe that this is a telescoping series with

$$s_n = \left(\frac{1}{1} - \frac{2}{2} + \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}\right).$$

The pattern here is slightly more complicated: given any three consecutive parenthetical groupings, the term in the middle with a coefficient of -2 is cancelled out by the combination of the last term in the preceding group and the first term in the following group. For instance, in the terms listed above, the $-\frac{2}{3}$ in the second group is cancelled out by the $+\frac{1}{3}$ in the first group and the $+\frac{1}{3}$ in the third group. Studying this carefully, we see that the only terms which fail to cancel out completely in this manner are the $\frac{1}{1}-\frac{2}{2}$ from the first group, the $\frac{1}{2}$ from the second group, the $\frac{1}{n}$ from the second-last group, and the $-\frac{2}{n}+\frac{1}{n+1}$ from the last group. Therefore, we can write

$$s_n = \frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} + \frac{1}{n+1} = \frac{1}{2} - \frac{1}{n} + \frac{1}{n+1},$$

and

$$\sum_{i=2}^{\infty} \frac{2}{i(i-1)(i+1)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}\right) = \frac{1}{2}.$$

5. (a) First we need to rewrite the given series as

$$\sum_{i=1}^{\infty} 4\left(\frac{2}{3}\right)^i = \frac{8}{3}\sum_{i=1}^{\infty}\left(\frac{2}{3}\right)^{i-1}.$$

This is a geometric series with $r = \frac{2}{3}$, so

$$\sum_{i=1}^{\infty} 4\left(\frac{2}{3}\right)^i = \frac{8}{3}\cdot\frac{1}{1-\frac{2}{3}} = \frac{8}{3}\cdot 3 = 8.$$

(b) Rewriting gives

$$\sum_{i=0}^{\infty}\frac{3^i-4^i}{3^i4^i} = \sum_{i=0}^{\infty}\frac{3^i}{3^i4^i} - \sum_{i=0}^{\infty}\frac{4^i}{3^i4^i} = \sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^i - \sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^i.$$

Both of these series are geometric; the first has $r = \frac{1}{4}$ and the second has $r = \frac{1}{3}$. So then

$$\sum_{i=0}^{\infty}\frac{3^i-4^i}{3^i4^i} = \frac{1}{1-\frac{1}{4}} - \frac{1}{1-\frac{1}{3}} = \frac{4}{3} - \frac{3}{2} = -\frac{1}{6}.$$

(c) Rewriting gives

$$\sum_{i=1}^{\infty}\left(\frac{2}{5}\right)^{3i} = \sum_{i=1}^{\infty}\left(\frac{8}{125}\right)^i = \frac{8}{125}\sum_{i=1}^{\infty}\left(\frac{8}{125}\right)^{i-1},$$

which is a geometric series with $r = \frac{8}{125}$. Then

$$\sum_{i=1}^{\infty}\left(\frac{2}{5}\right)^{3i} = \frac{8}{125}\cdot\frac{1}{1-\frac{8}{125}} = \frac{8}{117}.$$

(d) Rewriting gives

$$\sum_{i=1}^{\infty}(-1)^{i-1}(0.2)^{i-1} = \sum_{i=1}^{\infty}(-0.2)^{i-1},$$

which is a geometric series with $r = -0.2$. Then

$$\sum_{i=1}^{\infty}(-1)^{i-1}(0.2)^{i-1} = \frac{1}{1-(-0.2)} = \frac{1}{1.2} = \frac{5}{6}.$$

6. (a) Rewriting gives

$$\sum_{i=0}^{\infty}\frac{(x-6)^i}{4^i} = \sum_{i=0}^{\infty}\left(\frac{x-6}{4}\right)^i,$$

which is a geometric series with $r = \frac{x-6}{4}$. For this to converge, we require

$$-1 < \frac{x-6}{4} < 1 \quad \Longrightarrow \quad -4 < x-6 < 4 \quad \Longrightarrow \quad 2 < x < 10.$$

For these values of x,

$$\sum_{i=0}^{\infty}\frac{(x-6)^i}{4^i} = \frac{1}{1-\left(\frac{x-6}{4}\right)} = \frac{4}{10-x}.$$

(b) This is a geometric series with $r = \sin(x)$. It will converge when

$$-1 < \sin(x) < 1 \quad \Longrightarrow \quad x \neq \frac{(2k+1)\pi}{2},$$

where k is an integer. For such values of x,

$$\sum_{i=1}^{\infty}[\sin(x)]^{i-1} = \frac{1}{1-\sin(x)}.$$

7. (a) First observe that

$$\begin{aligned}
0.042424242\ldots &= 0.042 + 0.00042 + 0.0000042 + \cdots \\
&= 0.042(1 + 0.01 + 0.0001 + \cdots) \\
&= \sum_{i=1}^{\infty} 0.042(0.01)^{i-1} \\
&= \sum_{i=1}^{\infty}\frac{42}{1000}\left(\frac{1}{100}\right)^{i-1} \\
&= \frac{21}{500}\sum_{i=1}^{\infty}\left(\frac{1}{100}\right)^{i-1},
\end{aligned}$$

which is a geometric series with $r = \frac{1}{100}$. Then

$$0.042424242\ldots = \frac{21}{500}\sum_{i=1}^{\infty}\left(\frac{1}{100}\right)^{i-1} = \frac{21}{500}\cdot\frac{1}{1-\frac{1}{100}} = \frac{7}{165}.$$

(b) We can write

$$\begin{aligned}19.920920920\ldots &= 19 + 0.920 + 0.000920 + 0.000000920 + \cdots \\ &= 19 + 0.920(1 + 0.001 + 0.000001 + \cdots) \\ &= 19 + \sum_{i=1}^{\infty} 0.920(0.001)^{i-1} \\ &= 19 + \frac{23}{25}\sum_{i=1}^{\infty}\left(\frac{1}{1000}\right)^{i-1}\end{aligned}$$

which is a geometric series with $r = \frac{1}{1000}$. Then

$$19.920920920\ldots = 19 + \frac{23}{25}\sum_{i=1}^{\infty}\left(\frac{1}{1000}\right)^{i-1} = 19 + \frac{23}{25}\cdot\frac{1}{1-\frac{1}{1000}} = 19 + \frac{920}{999} = \frac{19901}{999}.$$

8. On the first bounce, the ball reaches a height of $1(0.6) = 0.6$. On the second bounce, the ball reaches a height of $(0.6)(0.6) = (0.6)^2$. On the third bounce, the ball reaches a height of $(0.6^2)(0.6) = (0.6)^3$. Continuing in this manner, it's clear that on the ball's ith bounce, it reaches a height of $(0.6)^i$. On each bounce, the ball travels up to its maximum height and then back down to the ground, for a distance travelled on the ith bounce of $2(0.6)^i$. Not forgetting the initial 1 metre drop, then, the ball's total distance travelled is

$$\begin{aligned}&1 + 2(0.6) + 2(0.6)^2 + 2(0.6)^3 + \cdots \\ &= 1 + 2(0.6)[1 + 0.6 + (0.6)^2 + \cdots] \\ &= 1 + 1.2\sum_{i=1}^{\infty}(0.6)^{i-1} \\ &= 1 + 1.2\cdot\frac{1}{1-0.6} \\ &= 1 + 3 \\ &= 4.\end{aligned}$$

So the ball travels a total of 4 metres.