Section 2.3

For functions of more than two variables, we again treat all variables with respect to which we are <u>not</u> differentiating as constants.

eg
$$f(x,y,z) = x\sin(z) - \cos(7y)$$
$$f_x(x,y,z) = \sin(z)$$
$$f_y(x,y,z) = 0 - [-\sin(7y)] \cdot 7$$
$$= 7\sin(7y)$$
$$f_z(x,y,z) = x\cos(z)$$

Second-order partial derivatives in prime notation for a function $z = f(x,y)$ are denoted by

$$z_{xx} = f_{xx}(x,y) \qquad z_{yx} = f_{yx}(x,y)$$
$$z_{xy} = f_{xy}(x,y) \qquad z_{yy} = f_{yy}(x,y)$$

The partial derivatives z_{xy} and z_{yx} are called <u>mixed</u> partial derivatives.

eg
$$f(x,y) = x^2 y$$
$$f_x(x,y) = 2xy \qquad f_y(x,y) = x^2$$
$$f_{xx}(x,y) = 2y \qquad f_{yy}(x,y) = 0$$
$$f_{xy}(x,y) = 2x \qquad f_{yx}(x,y) = 2x$$

Clairault's Theorem

If $f(x,y)$ is defined in an open circle D of a point (p,q) and both $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are continuous in D then

$$f_{xy}(p,q) = f_{yx}(p,q)$$

Leibniz notation for second-order partial derivatives has the form

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}, \quad \frac{\partial^2 z}{\partial y^2}$$

A <u>partial differential equation</u> (PDE) is an equation involving an unknown function of two or more variables and one or more of its partial derivatives. A function which satisfies the PDE is said to be a <u>solution</u>.

To check whether a function is a solution of a PDE, we compute the relevant partial derivatives, substitute them into the equation and determine whether the left-hand and right-hand sides are equal.

eg $[f_x(x,y)]^2 = f_{xx}(x,y)\, f_{xy}(x,y)$

Is $f(x,y) = x^2 y$ a solution of this PDE?

$f_x(x,y) = 2xy$

$f_{xx}(x,y) = 2y \qquad f_{xy}(x,y) = 2x$

On the left: $[f_x(x,y)]^2 = (2xy)^2 = 4x^2y^2$

On the right: $f_{xx}(x,y)\, f_{xy}(x,y) = 2y \cdot 2x = 4xy$

These are not equal, so $f(x,y) = x^2 y$ is <u>not</u> a solution.

Laplace's equation is given by

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

eg Determine whether $z = \ln(x^2+y^2)$ is a solution to Laplace's equation.

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2(x^2+y^2) - 2y(2y)}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

Left-hand side: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{y^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2}$

$$= \frac{0}{(x^2+y^2)^2} = 0$$

Thus $z = \ln(x^2+y^2)$ <u>is</u> a solution of Laplace's equation.