Section 2.7

eg Find the volume under the hemisphere $z = \sqrt{1-x^2-y^2}$.

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} \, dy \, dx$$

$$= \int_{-1}^{1} \int_{-\pi/2}^{\pi/2} (1-x^2) \cos^2(t) \, dt \, dx$$

$$= \int_{-1}^{1} \int_{-\pi/2}^{\pi/2} (1-x^2) \cdot \frac{1+\cos(2t)}{2} \, dt \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} \int_{-\pi/2}^{\pi/2} (1-x^2) [1 + \cos(2t)] \, dt \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left[(1-x^2) \left(t + \frac{1}{2} \sin(2t) \right) \right]_{t=-\pi/2}^{t=\pi/2} dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1-x^2) \pi \, dx$$

$$= \frac{\pi}{2} \int_{-1}^{1} (1-x^2) \, dx$$

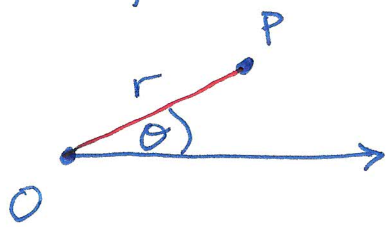
$$= \frac{\pi}{2} \left[x - \frac{1}{3} x^3 \right]_{-1}^{1}$$

$$\boxed{= \frac{2\pi}{3}}$$

Section 2.8: Polar Coordinates

The xy-coordinate system is properly known as the Cartesian or rectangular coordinate system. It requires two perpendicular axes which meet at the origin O. The perpendicular distance from the vertical axis measures the x-coordinate, and the perpendicular distance from the horizontal axis measures the y-coordinate.

In the polar coordinate system we choose a reference point called the origin (or pole) denoted by O and a half-line drawn starting from O called the polar axis. By convention, we normally choose the Cartesian and polar origins to be the same point, and identify the polar axis with the positive x-axis.



Let r be the straight-line distance from O to a point P.

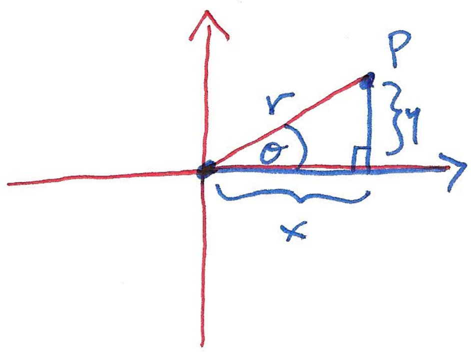
Let θ be the angle that the line OP makes with the polar axis. By convention we will take θ to be positive if it is measured counter-clockwise.

The coordinates (r, θ) are then called the polar coordinates of P.

Note that (r, θ) and $(r, \theta + 2\pi k)$ for any integer k must refer to the same point.

If $r < 0$ then we define $(r, \theta) = (-r, \theta + \pi)$.

Also, the polar origin can be represented by $(0, \theta)$ for any angle θ.



By the Pythagorean theorem,

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

From trigonometry,

$$\tan(\theta) = \frac{y}{x}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

Again from trigonometry,

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

eg Convert the point $(\sqrt{3}, 1)$ from Cartesian to polar coordinates.

$$r = \sqrt{3+1} = 2$$

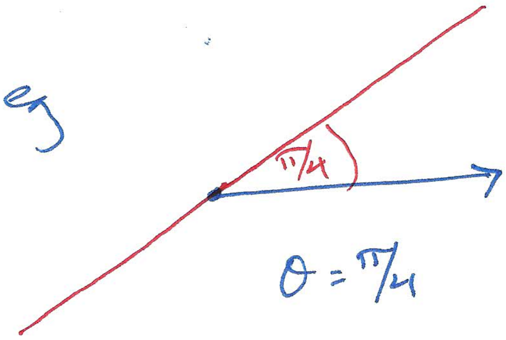
$$\theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

In polar coordinates, this is the point $(2, \pi/6)$.

In Cartesian coordinates $y=k$ and $x=k$ (for any constant k) are represented graphically by horizontal and vertical lines, respectively.

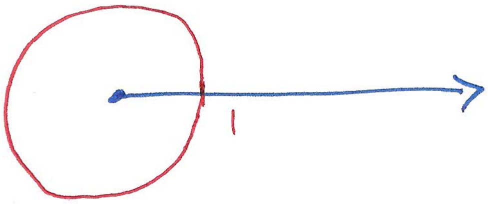
Likewise, $\theta = k$ has a line as its graph but it is specifically the line through the origin which makes an angle k with the polar axis.

eg



$\theta = \pi/4$

eg



$r = 1$

On the other hand, the graph of $r=k$ is a circle, centred at the origin, of radius k.

eg The unit circle in Cartesian coordinates has equation

$$x^2 + y^2 = 1$$

$$[r\cos(\theta)]^2 + [r\sin(\theta)]^2 = 1$$

$$r^2\cos^2(\theta) + r^2\sin^2(\theta) = 1$$

$$r^2\left[\cos^2(\theta)+\sin^2(\theta)\right]=1$$

$$r^2\cdot 1=1$$

$$r^2=1$$

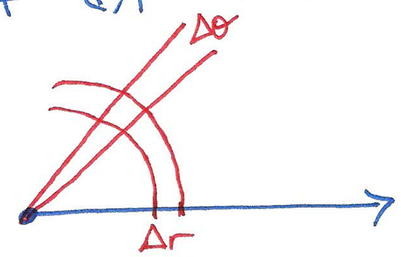
$$r=1$$

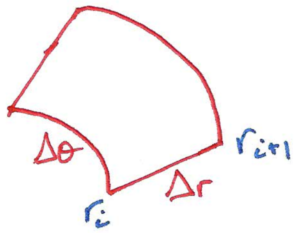
Section 2.9: Double Integrals in Polar Coordinates

In polar coordinates, a double integral has the form

$$\iint_D f(r,\theta)\,dA$$

However, whereas $dA=dx\,dy$ in Cartesian coordinates, it is not true that $dA=dr\,d\theta$ in polar coordinates.





In polar coordinates a region is divided into a number of small polar rectangles. The area of a polar rectangle is given by

$$\Delta A=\frac{1}{2}\Delta\theta\,\Delta r\,(r_i+r_{i+1})$$

As the number of polar rectangles becomes infinitely large, we see that

$$dA=d\theta\,dr\cdot r=r\,dr\,d\theta$$

Thus

$$\iint_D f(r,\theta)\,dA=\int_\alpha^\beta\int_a^b f(r,\theta)\,r\,dr\,d\theta$$

$$=\int_a^b\int_\alpha^\beta f(r,\theta)\,r\,d\theta\,dr$$