

Section 1.7

As an alternative to solving a quadratic equation by factoring we can instead use the quadratic formula.

Given the equation $ax^2 + bx + c = 0$ for $a \neq 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here the expression $b^2 - 4ac$ is the discriminant.

eg Solve $2x^2 - 2x - 3 = 0$.

We can easily see that $2x^2 - 2x - 3$ is not factorable, so we use the quadratic formula instead:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 2 \cdot (-3)}}{2 \cdot 2}$$

$$= \frac{2 \pm \sqrt{4 + 24}}{4}$$

$$= \frac{2 \pm \sqrt{28}}{4}$$

$$= \frac{2 \pm 2\sqrt{7}}{4}$$

$$= \frac{2(1 \pm \sqrt{7})}{4}$$

$$= \frac{1 \pm \sqrt{7}}{2}$$

There are two solutions:

$$x = \frac{1+\sqrt{7}}{2} \text{ and } x = \frac{1-\sqrt{7}}{2}$$

eg Solve $x^2 + 2x + 3 = 0$.

We use the quadratic formula:

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1}$$

$$= \frac{-2 \pm \sqrt{4 - 12}}{2}$$

$$= \frac{-2 \pm \sqrt{-8}}{2}$$

Since $\sqrt{-8}$ is undefined, this equation has no solutions.

The quadratic formula indicates that there are 3 possible cases for solutions of a quadratic equation:

① if $b^2 - 4ac > 0$ then there are 2 distinct solutions

② if $b^2 - 4ac < 0$ then the equation has no solutions

③ if $b^2 - 4ac = 0$ then the equation has exactly one solution, called a repeated solution or a solution of multiplicity 2

We can use factoring to solve polynomial equations involving powers of x larger than 2.

eg Solve $x^3 - 5x^2 + 6x = 0$.

We have $x(x^2 - 5x + 6) = 0$

$$x(x-2)(x-3) = 0$$

Thus either $x=0$ or $x-2=0$ or $x-3=0$
 $x=2$ $x=3$

There are 3 different solutions:

$x=0$
$x=2$
$x=3$

eg Solve $2x^3 - x^2 - 4x + 2 = 0$.

We have $2x^3 - x^2 - 4x + 2 = x^2(2x-1) - 2(2x-1)$
 $= (2x-1)(x^2-2)$

We now have

$$(2x-1)(x^2-2) = 0$$

So $2x-1=0$

$$2x=1$$

$$x = \frac{1}{2}$$

or

$$x^2-2=0$$

$$x^2=2$$

$$x = \pm\sqrt{2}$$

Thus the solutions are $\boxed{x = \frac{1}{2}, x = \sqrt{2}, x = -\sqrt{2}}$.

eg Solve $x^3 - x^2 = 0$.

We have $x^2(x-1) = 0$

$$x^2=0$$

$$x=0$$

or

$$x-1=0$$

$$x=1$$

The solutions are $\boxed{x=0, x=1}$.

Section 1.8: Non-Polynomial Equations

The strategies for solving polynomial equations can be adapted for certain types of non-polynomial equations.

A linear absolute value equation involves the absolute value of a linear polynomial. To solve it, we use the fact that if $|x| = k$ then $x = k$ or $x = -k$.

eg Solve $|x-3| = 4$.

We could have $x-3=4$ or $x-3=-4$
 $x=7$ $x=-1$

There are 2 solutions: $x=7$ and $x=-1$.

In general, we must verify that any solution obtained for an absolute value equation actually satisfies the given equation. This is because our solution method can produce spurious solutions which must be omitted.

eg Solve $|x+2| + 3x = 2$.

First we isolate the absolute value by rewriting the equation as

$$|x+2| = 2-3x$$

Now we can have

$$x+2 = 2-3x \quad \text{or} \quad x+2 = -(2-3x)$$

$$4x = 0$$

$$x = 0$$

$$x+2 = -2+3x$$

$$4 = 2x$$

$$2 = x$$

$$\text{For } x=0, \quad |x+2| + 3x = |0+2| + 3 \cdot 0$$
$$= 2+0$$

$= 2$ so $x=0$ is a solution

$$\text{For } x=2, \quad |x+2| + 3x = |2+2| + 3 \cdot 2$$
$$= 4+6$$

$= 10 \neq 2$ so $x=2$ is not a solution

Thus only $\boxed{x=0}$ is a solution.

Why is $x=2$ a spurious solution? Recall that

$$|x+2| = \begin{cases} x+2 & \text{for } x+2 \geq 0 \text{ so } x \geq -2 \\ -(x+2) & \text{for } x+2 < 0 \text{ so } x < -2 \end{cases}$$

Both $x=0$ and $x=2$ use $|x+2| = x+2$.

However, we only obtain $x=2$ as a solution if

$$|x+2| = -(x+2).$$