

Section 2.2: Functions

Def'n: A function is a rule which assigns each element of a set D , called the domain, to an element of a set R , called the range.

This rule may be a formula, but may take other forms as well.

If we label the function f and $x \in D$ then the corresponding element in R is denoted by $f(x)$, which is the image of x .

eg Suppose f is a function where the rule is that any real number is doubled and added to its own square, and then 3 is subtracted from the result. Then

$$f(x) = x^2 + 2x - 3$$

To evaluate a function, we choose a value of x in D and substitute that value in place of every instance of x in the rule.

eg For $f(x) = x^2 + 2x - 3$ we have

$$f(0) = 0^2 + 2 \cdot 0 - 3 = 0 + 0 - 3 = -3$$

$$f(2) = 2^2 + 2 \cdot 2 - 3 = 4 + 4 - 3 = 5$$

$$f(-\frac{1}{3}) = (-\frac{1}{3})^2 + 2 \cdot (-\frac{1}{3}) - 3$$

$$= \frac{1}{9} - \frac{2}{3} - 3$$

$$= \frac{1}{9} - \frac{6}{9} - \frac{27}{9}$$

$$= -\frac{32}{9}$$

We can also evaluate a function using a mathematical expression, by replacing every instance of x with that expression, and then simplifying the result.

eg For $f(x) = x^2 + 2x - 3$ we have

$$f(x^2 + 1) = (x^2 + 1)^2 + 2(x^2 + 1) - 3$$

$$= x^4 + 2x^2 + 1 + 2x^2 + 2 - 3$$

$$= x^4 + 4x^2$$

Unless otherwise specified, we assume that the domain of a function consists of all real numbers for which the rule is mathematically valid.

eg For $f(x) = x^2 + 2x - 3$ the domain is $D = \mathbb{R}$
because any real number x can be substituted into
 $f(x)$ to obtain another real number.

eg For $f(x) = \frac{x+1}{x^2-1}$ we must determine when
division by zero occurs, by setting

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm\sqrt{1} = \pm 1$$

Thus $D = \{x \mid x \neq \pm 1\}$.

eg For $f(x) = \sqrt{x-3}$ we must have $x-3 \geq 0$ so
 $x \geq 3$. Thus $D = \{x \mid x \geq 3\}$ which is
equivalent to the interval $[3, \infty)$.

eg For $f(x) = \frac{\sqrt{4-x}}{x}$ we require both $4-x \geq 0$
so $x \leq 4$, and $x \neq 0$. Thus $D = \{x \mid x \leq 4, x \neq 0\}$.

eg For $f(x) = \frac{5x}{x^2+9}$ we must have $x^2+9 \neq 0$.

But if we set $x^2+9=0$ we obtain $x^2 = -9$
which has no solutions. Thus $D = \mathbb{R}$.

In some cases, we will choose to define a function such that its domain omits some values of x that would produce a mathematically valid value of $f(x)$. This is called restricting the domain of the function.

eg Consider $g(x) = \frac{5x}{x^2+9}$ for $x \geq 0$.

This function has a restricted domain because it excludes $x < 0$, even though $\frac{5x}{x^2+9}$ is defined there.

To determine the range of a function, we have to apply our familiarity with how the rule of the function behaves.

eg Consider $f(x) = x^2 + 2$.

We know that $x^2 \geq 0$ so $x^2 + 2 \geq 0 + 2$
 $x^2 + 2 \geq 2$

Thus the range of $f(x)$ is $R = [2, \infty)$.

Most often, the range of a function is more easily determined by visualizing a function, that is, by sketching its graph.

Since we often use x to represent the elements of the domain, we often use y to represent the elements of the range. Thus we often denote a function in the form $y = f(x)$.

eg If $f(x) = x^2 + 2$ then, for any $y \in \mathbb{R}$, we know that $y = x^2 + 2$.

If a function follows different rules for different subsets of its domain, we call it a piecewise function.

eg Consider a function f where, for $x < 1$ the rule is that x is halved, for $x > 1$ the rule is that x is tripled, and for $x = 1$ the rule is that the image is 7. This is a piecewise function, and can be denoted by

$$f(x) = \begin{cases} x/2, & \text{for } x < 1 \\ 7, & \text{for } x = 1 \\ 3x, & \text{for } x > 1. \end{cases}$$

Thus, say, $f(0) = 0/2 = 0$

$$f(-4) = -4/2 = -2$$

$$f(4) = 3 \cdot 4 = 12$$

$$f(1) = 7$$

When the rule of a function involves an absolute value, this is a special kind of piecewise function.

For instance, given $f(x) = |x|$ we can write this as

$$f(x) = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases}$$