

Section 2.1: Solving Inequalities

We can manipulate and solve inequalities much like equations. But the two processes are not identical.

Theorem: Properties of Inequalities

Suppose b, c, d are real numbers or mathematical expressions. Then

- ① if $b < c$ then $b+d < c+d$ and $b-d < c-d$
- ② if $b < c$ and $d > 0$ then $b \cdot d < c \cdot d$ and $\frac{b}{d} < \frac{c}{d}$
- ③ if $b < c$ and $d < 0$ then $b \cdot d > c \cdot d$ and $\frac{b}{d} > \frac{c}{d}$

eg We know that $-2 < 3$ and so

$$-2.5 < 3.5 \text{ because } -10 < 15$$

$$\text{but } -2 \cdot (-5) > 3 \cdot (-5) \text{ because } 10 > -15$$

This theorem also applies to non-strict inequalities.

Def'n: A linear inequality is an inequality that can be written in the form

$$ax + b < 0 \quad \text{or} \quad ax + b \leq 0$$

where a, b are real numbers and $a \neq 0$.

eg Solve $7(x-1) < 5x-13$.

$$7x-7 < 5x-13$$

$$7x-5x < -13+7$$

$$2x < -6$$

$$\frac{2x}{2} < \frac{-6}{2}$$

$$x < -3$$

The solution of this linear inequality is
the interval $\boxed{(-\infty, -3)}$.

eg Solve $\frac{x}{3} \geq \frac{x}{2} + 1$

$$\frac{x}{3} - \frac{x}{2} \geq 1$$

$$\frac{x}{3} \cdot \frac{2}{2} - \frac{x}{2} \cdot \frac{3}{3} \geq 1$$

$$\frac{2x}{6} - \frac{3x}{6} \geq 1$$

$$\frac{-x}{6} \geq 1$$

$$-\frac{1}{6}x \geq 1$$

$$-\frac{1}{6}x \cdot (-6) \leq 1 \cdot (-6)$$

$$x \leq -6$$

The solution is $\boxed{(-\infty, -6]}$.

Many inequalities incorporate two simultaneous conditions, which is called a double inequality. Sometimes, a double inequality can be solved just like a single inequality.

eg Solve $-7 < 3x - 1 < 5$.

$$\begin{aligned} -7 + 1 &< 3x < 5 + 1 \\ -6 &< 3x < 6 \\ -\frac{6}{3} &< \frac{3x}{3} < \frac{6}{3} \\ -2 &< x < 2 \end{aligned}$$

The solution is $\boxed{(-2, 2)}$.

When the variable appears on more than one side of a double inequality, it's usually easier to treat it as two single inequalities, solve them individually, and combine the results.

eg Solve $x + 3 \leq 4x - 3 < 2x + 7$.

First we solve $x + 3 \leq 4x - 3$

$$3 + 3 \leq 4x - x$$

$$6 \leq 3x$$

$$\frac{6}{3} \leq \frac{3x}{3}$$

$$2 \leq x$$

Next we solve $4x - 3 < 2x + 7$

$$4x - 2x < 7 + 3$$

$$2x < 10$$

$$\frac{2x}{2} < \frac{10}{2}$$

$$x < 5$$

To satisfy the double inequality, we must combine these conditions to obtain $2 \leq x < 5$. The solution is $\boxed{[2, 5)}$.

Many inequalities involve absolute values, for which we have the following result:

Theorem: Properties of Absolute Value Inequalities

Consider $b > 0$. Then

① if $|x| < b$ then $-b < x < b$

② if $|x| > b$ then $x > b$ or $x < -b$

Again, we can replace the strict inequalities in this result with non-strict inequalities.

eg Solve $|3-2x| < 5$.

We can use Property #1 to rewrite this inequality as the double inequality

$$-5 < 3-2x < 5$$

$$-5-3 < -2x < 5-3$$

$$-8 < -2x < 2$$

$$\frac{-8}{-2} > \frac{-2x}{-2} > \frac{2}{-2}$$

$$4 > x > -1$$

The solution is $\boxed{(-1, 4)}$.

eg Solve $|5x+1| \geq 6$.

We can have $5x+1 \geq 6$ or $5x+1 \leq -6$.

First, $5x+1 \geq 6$ becomes

$$5x \geq 6-1$$

$$5x \geq 5$$

$$\frac{5x}{5} \geq \frac{5}{5}$$

$$x \geq 1$$

Next, $5x+1 \leq -6$ becomes $5x \leq -6-1$

$$5x \leq -7$$

$$\frac{5x}{5} \leq \frac{-7}{5}$$

$$x \leq -\frac{7}{5}$$

Since only one of these conditions must be satisfied by a solution of the inequality, we can have $x \geq 1$ or $x \leq -\frac{7}{5}$. These are the intervals $[1, \infty)$ and $(-\infty, -\frac{7}{5}]$. So the solution is $\boxed{(-\infty, -\frac{7}{5}] \cup [1, \infty)}$.