

Section 1.8: Non-Polynomial Equations

The strategies for solving polynomial equations can be adapted for certain types of non-polynomial equations.

A linear absolute value equation involves the absolute value of a linear polynomial. To solve it, we use the fact that if $|x| = k$ then $x = k$ or $x = -k$.

eg Solve $|x-3| = 4$.

We could have $x-3=4$ or $x-3=-4$
 $x=7$ $x=-1$

There are 2 solutions: $x=7$ and $x=-1$.

In general, we must verify that any solution obtained for an absolute value equation actually satisfies the given equation. This is because our solution method can produce spurious solutions which must be omitted.

eg Solve $|x+2| + 3x = 2$.

First we isolate the absolute value by rewriting the equation as

$$|x+2| = 2-3x$$

Now we can have

$$\begin{aligned}x+2 &= 2-3x & \text{or} & & x+2 &= -(2-3x) \\4x &= 0 & & & x+2 &= -2+3x \\x &= 0 & & & 4 &= 2x \\& & & & 2 &= x\end{aligned}$$

$$\text{For } x=0, \quad |x+2|+3x = |0+2|+3\cdot 0$$

$$= 2+0$$

$= 2$ so $x=0$ is a solution

$$\text{For } x=2, \quad |x+2|+3x = |2+2|+3\cdot 2$$

$$= 4+6$$

$= 10 \neq 2$ so $x=2$ is not a solution

Thus only $\boxed{x=0}$ is a solution.

Why is $x=2$ a spurious solution? Recall that

$$|x+2| = \begin{cases} x+2 & \text{for } x+2 \geq 0 \text{ so } x \geq -2 \\ -(x+2) & \text{for } x+2 < 0 \text{ so } x < -2 \end{cases}$$

Both $x=0$ and $x=2$ use $|x+2| = x+2$.

However, we only obtain $x=2$ as a solution if

$$|x+2| = -(x+2).$$

To solve a rational equation we write all rational expressions over the lowest common denominator.

eg Solve $\frac{2}{x-1} + \frac{x}{x-2} = \frac{x-3}{x^2-3x+2}$.

We have $x^2-3x+2 = (x-1)(x-2)$.

This is the lowest common denominator, so we write the equation as

$$\frac{2}{x-1} \cdot \frac{x-2}{x-2} + \frac{x}{x-2} \cdot \frac{x-1}{x-1} = \frac{x-3}{(x-1)(x-2)}$$

$$\frac{2x-4}{(x-1)(x-2)} + \frac{x^2-x}{(x-1)(x-2)} = \frac{x-3}{(x-1)(x-2)}$$

$$\frac{x^2+x-4}{(x-1)(x-2)} = \frac{x-3}{(x-1)(x-2)}$$

This can only be true if the numerators are equal, so

$$x^2+x-4 = x-3$$

$$x^2-1 = 0$$

$$(x-1)(x+1) = 0$$

$$x-1=0 \quad \text{or} \quad x+1=0$$

$$x=1$$

$$x=-1$$

Now observe that $\frac{2}{x-1}$ is undefined for $x=1$

$\frac{x}{x-2}$ is undefined for $x=2$

$\frac{x-3}{x^2-3x+2}$ is undefined for $x=1$ and $x=2$

Thus $x=1$ is a spurious solution so the only solution is $\boxed{x=-1}$.

eg Solve $\frac{3}{x} - \frac{2}{x+2} - \frac{1}{x-2} = 0$.

The lowest common denominator is $x(x+2)(x-2)$

so we write

$$\frac{3}{x} \cdot \frac{(x+2)(x-2)}{(x+2)(x-2)} - \frac{2}{x+2} \cdot \frac{x(x-2)}{x(x-2)} - \frac{1}{x-2} \cdot \frac{x(x+2)}{x(x+2)} = 0$$

$$\frac{3(x^2-4)}{x(x+2)(x-2)} - \frac{2(x^2-2x)}{x(x+2)(x-2)} - \frac{(x^2+2x)}{x(x+2)(x-2)} = 0$$

$$\frac{3x^2-12 - (2x^2-4x) - (x^2+2x)}{x(x+2)(x-2)} = 0$$

$$\frac{2x-12}{x(x+2)(x-2)} = 0$$

Now we see that $2x-12=0$

$$2x=12$$

$$x=6$$

Observe that $\frac{3}{x}$ is undefined for $x=0$

$\frac{2}{x+2}$ is undefined for $x=-2$

$\frac{1}{x-2}$ is undefined for $x=2$

Thus $\boxed{x=6}$ is the only solution.

Given a radical equation we first isolate the radical.
We can then remove the radical by raising both sides of the equation to the power of n , where n is the index of the radical.

eg Solve $\sqrt{3x+7} = x+1$.

The square root is already isolated, so we have

$$(\sqrt{3x+7})^2 = (x+1)^2$$

$$3x+7 = x^2+2x+1$$

$$0 = x^2 - x - 6$$

$$0 = (x-3)(x+2)$$

The possible solutions are $x-3=0$ so $x=3$
 $x+2=0$ so $x=-2$

These solutions could satisfy $\sqrt{3x+7} = x+1$ or they could satisfy $\sqrt{3x+7} = -(x+1)$. We must substitute back into the given equation to check its validity.

For $x=3$, we have $\sqrt{3x+7} = \sqrt{16} = 4$
 $x+1 = 3+1 = 4$ so $x=3$ is a solution

For $x=-2$, we have $\sqrt{3x+7} = \sqrt{1} = 1$
 $x+1 = -2+1 = -1$ so $x=-2$ is not a solution

The only solution is $\boxed{x=3}$.

eg Solve $\sqrt[3]{x^2-8} - 2 = 0$.

First we write

$$\sqrt[3]{x^2-8} = 2$$

$$(\sqrt[3]{x^2-8})^3 = 2^3$$

$$x^2-8 = 8$$

$$x^2 = 16$$

$$x = \pm\sqrt{16} = \pm 4$$

For $x=4$, we have $\sqrt[3]{8} - 2 = 2 - 2 = 0$.

For $x=-4$, we have $\sqrt[3]{8} - 2 = 2 - 2 = 0$.

Both $\boxed{x=4 \text{ and } x=-4}$ are solutions.

If more than one radical is present in the equation, we choose radical to isolate and remove, and then repeat the process with each radical that remains.

eg Solve $3 - \sqrt{x+4} = \sqrt{5-x}$.

We have

$$(3 - \sqrt{x+4})^2 = (\sqrt{5-x})^2$$

$$(3 - \sqrt{x+4})(3 - \sqrt{x+4}) = 5 - x$$

$$9 - 3\sqrt{x+4} - 3\sqrt{x+4} + x + 4 = 5 - x$$

$$x + 13 - 6\sqrt{x+4} = 5 - x$$

$$2x + 8 = 6\sqrt{x+4}$$

$$x + 4 = 3\sqrt{x+4}$$

$$(x+4)^2 = (3\sqrt{x+4})^2$$

$$x^2 + 8x + 16 = 9(x+4)$$

$$x^2 + 8x + 16 = 9x + 36$$

$$x^2 - x - 20 = 0$$

$$(x-5)(x+4) = 0$$

Thus $x-5=0$ so $x=5$

or $x+4=0$ so $x=-4$

For $x=5$, we have $3 - \sqrt{x+4} = 3 - \sqrt{9} = 3 - 3 = 0$
 $\sqrt{5-x} = \sqrt{0} = 0$

So $x=5$ is a solution.

For $x=-4$, we have $3 - \sqrt{x+4} = 3 - \sqrt{0} = 3 - 0 = 3$
 $\sqrt{5-x} = \sqrt{9} = 3$

So $x=-4$ is also a solution.

Both $x=5$ and $x=-4$ are solutions.