

Section 1.7: Solving Equations

Def'n: An equation is a mathematical statement in which two expressions are set equal to each other. If the equation includes a variable, a solution of the equation is a value of the variable that makes the equation true. To solve the equation is to find all solutions.

Def'n: A linear equation is an equation that can be written in the form

$$ax + b = 0$$

for real numbers a, b with $a \neq 0$.

eg $4x - 6 = 0$ is a linear equation

To solve a linear equation, we just isolate x :

$$ax + b = 0$$

$$ax = -b$$

$$\frac{ax}{a} = -\frac{b}{a}$$

$$x = -\frac{b}{a}$$

eg Solve $4x - 6 = 0$.

We have $4x - 6 = 0$

$$4x = 6$$

$$\frac{4x}{4} = \frac{6}{4}$$

$$\boxed{x = \frac{3}{2}}$$

We can check our solutions by substituting back into the equation.

eg Here, we see that for $x = \frac{3}{2}$,

$$4x - 6 = 4 \cdot \frac{3}{2} - 6 = 6 - 6 = 0.$$

eg Solve $\frac{x}{6} = 1 - \frac{5x}{9}$

We can rewrite this equation as

$$\frac{x}{6} + \frac{5x}{9} = 1$$

$$\frac{x}{6} \cdot \frac{3}{3} + \frac{5x}{9} \cdot \frac{2}{2} = 1$$

$$\frac{3x}{18} + \frac{10x}{18} = 1$$

$$\frac{13x}{18} = 1$$

$$\frac{13x}{18} \cdot \frac{18}{13} = 1 \cdot \frac{18}{13}$$

$$\boxed{x = \frac{18}{13}}$$

Def'n: A quadratic equation is an equation that can be written in the form

$$ax^2 + bx + c = 0$$

for real numbers a, b, c with $a \neq 0$.

eg $3x^2 - 9 = 0$ and $x^2 - 6x + 8 = 0$
are both quadratic equations

The easiest type of quadratic equation to solve is one in which $b = 0$ so we have only $ax^2 + c = 0$. They can be solved in a manner similar to linear equations.

eg Solve $3x^2 - 9 = 0$.

We have

$$3x^2 - 9 = 0$$

$$3x^2 = 9$$

$$\frac{3x^2}{3} = \frac{9}{3}$$

$$x^2 = 3$$

$$\boxed{x = \pm \sqrt{3}}$$

If $b \neq 0$ we can try to solve the quadratic equation by factoring the quadratic polynomial.

eg Solve $x^2 - 6x + 8 = 0$.

We can factor $x^2 - 6x + 8 = (x-2)(x-4)$.

Thus the equation becomes

$$(x-2)(x-4) = 0$$

This means that $x-2=0$ or $x-4=0$

so the solutions are $\boxed{x=2 \text{ and } x=4}$.

eg Solve $(x-3)(x-1) = -1$.

First we must expand the left hand side:

$$\begin{aligned}(x-3)(x-1) &= x(x-1) - 3(x-1) \\ &= x^2 - x - 3x + 3 \\ &= x^2 - 4x + 3\end{aligned}$$

The equation becomes

$$x^2 - 4x + 3 = -1$$

$$x^2 - 4x + 4 = 0$$

$$(x-2)^2 = 0$$

Hence $x-2=0$ so the only solution is $\boxed{x=2}$.

As an alternative to solving a quadratic equation by factoring we can instead use the quadratic formula.

Given the equation $ax^2 + bx + c = 0$ for $a \neq 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here the expression $b^2 - 4ac$ is the discriminant.

eg Solve $2x^2 - 2x - 3 = 0$.

We can easily see that $2x^2 - 2x - 3$ is not factorable, so we use the quadratic formula instead:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 2 \cdot (-3)}}{2 \cdot 2}$$

$$= \frac{2 \pm \sqrt{4 + 24}}{4}$$

$$= \frac{2 \pm \sqrt{28}}{4}$$

$$= \frac{2 \pm 2\sqrt{7}}{4}$$

$$= \frac{2(1 \pm \sqrt{7})}{4}$$

$$= \frac{1 \pm \sqrt{7}}{2}$$

There are two solutions: $x = \frac{1+\sqrt{7}}{2}$ and $x = \frac{1-\sqrt{7}}{2}$

eg Solve $x^2 + 2x + 3 = 0$.

We use the quadratic formula:

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1}$$

$$= \frac{-2 \pm \sqrt{4 - 12}}{2}$$

$$= \frac{-2 \pm \sqrt{-8}}{2}$$

Since $\sqrt{-8}$ is undefined, this equation has no solutions.

The quadratic formula indicates that there are 3 possible cases for solutions of a quadratic equation:

① if $b^2 - 4ac > 0$ then there are 2 distinct solutions

② if $b^2 - 4ac < 0$ then the equation has no solutions

③ if $b^2 - 4ac = 0$ then the equation has exactly one solution, called a repeated solution or a solution of multiplicity 2

We can use factoring to solve polynomial equations involving powers of x larger than 2.

eg Solve $x^3 - 5x^2 + 6x = 0$.

We have $x(x^2 - 5x + 6) = 0$

$$x(x-2)(x-3) = 0$$

Thus either $x=0$ or $x-2=0$ or $x-3=0$
 $x=2$ $x=3$

There are 3 different solutions:

$x=0$
$x=2$
$x=3$

eg Solve $2x^3 - x^2 - 4x + 2 = 0$.

We have $2x^3 - x^2 - 4x + 2 = x^2(2x-1) - 2(2x-1)$
 $= (2x-1)(x^2-2)$

We now have

$$(2x-1)(x^2-2) = 0$$

So $2x-1=0$

or

$$x^2-2=0$$

$$2x=1$$

$$x^2=2$$

$$x=\frac{1}{2}$$

$$x=\pm\sqrt{2}$$

Thus the solutions are $\boxed{x=\frac{1}{2}, x=\sqrt{2}, x=-\sqrt{2}}$.

eg Solve $x^3 - x^2 = 0$.

We have $x^2(x-1) = 0$

$$x^2=0$$

or

$$x-1=0$$

$$x=0$$

$$x=1$$

The solutions are $\boxed{x=0, x=1}$.