

Section 1.5

We can multiply or divide rational expressions exactly like we multiply or divide fractions.

eg Divide $\frac{x^3+27}{x^2-4}$ by $\frac{x^2-3x+9}{12-6x}$.

Dividing by fraction is the same as multiplying by its reciprocal, so dividing by a rational expression works the same way. Here we have

$$\frac{x^3+27}{x^2-4} \cdot \frac{12-6x}{x^2-3x+9}$$
$$= \frac{(x+3)(x^2-3x+9)}{(x-2)(x+2)} \cdot \frac{-6(x-2)}{x^2-3x+9}$$

$$\boxed{= \frac{-6(x+3)}{x+2}}$$

To add or subtract rational expressions, they must be written over a common denominator. It is usually most efficient to use the lowest common denominator, which can be obtained by factoring.

$$\begin{aligned}
 \text{eg } \frac{4}{x^2-1} - \frac{2}{x^2+x} &= \frac{4}{(x-1)(x+1)} - \frac{2}{x(x+1)} \\
 &= \frac{4}{(x-1)(x+1)} \cdot \frac{x}{x} - \frac{2}{x(x+1)} \cdot \frac{x-1}{x-1} \\
 &= \frac{4x}{x(x-1)(x+1)} - \frac{2(x-1)}{x(x-1)(x+1)} \\
 &= \frac{4x - 2(x-1)}{x(x-1)(x+1)} \\
 &= \frac{4x - (2x - 2)}{x(x-1)(x+1)} \\
 &= \frac{4x - 2x + 2}{x(x-1)(x+1)} \\
 &= \frac{2x + 2}{x(x-1)(x+1)} \\
 &= \frac{2(x+1)}{x(x-1)(x+1)} \\
 &= \boxed{\frac{2}{x(x-1)}}
 \end{aligned}$$

We can also write rational expressions in terms of negative exponents.

eg $(x-3)(x+1)^{-3} + x(x+1)^{-2}$

We can rewrite this as

$$\begin{aligned} \frac{x-3}{(x+1)^3} + \frac{x}{(x+1)^2} &= \frac{x-3}{(x+1)^3} + \frac{x}{(x+1)^2} \cdot \frac{x+1}{x+1} \\ &= \frac{x-3}{(x+1)^3} + \frac{x(x+1)}{(x+1)^3} \\ &= \frac{(x-3) + (x^2+x)}{(x+1)^3} \\ &= \frac{x^2+2x-3}{(x+1)^3} \\ &= \boxed{\frac{(x-1)(x+3)}{(x+1)^3}} \end{aligned}$$

A complex rational expression is an expression that includes a rational expression in its numerator, its denominator, or both.

eg $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$ is a complex rational expression

Here we can write $1 + \frac{1}{x} = \frac{x}{x} + \frac{1}{x} = \frac{x+1}{x}$

and $1 - \frac{1}{x} = \frac{x}{x} - \frac{1}{x} = \frac{x-1}{x}$

Now we can write

$$\frac{1 + 1/x}{1 - 1/x} = \frac{\left(\frac{x+1}{x}\right)}{\left(\frac{x-1}{x}\right)} = \frac{x+1}{x} \cdot \frac{x}{x-1}$$

$$= \frac{x+1}{x-1}$$

eg $\frac{2}{1 + (x+3)^{-1}}$

A different approach is to remove the internal rational expression by multiplying the numerator and the denominator of the entire expression by the denominator of the internal rational expression.

Here we write

$$\begin{aligned}\frac{2}{1 + (x+3)^{-1}} &= \frac{2}{1 + 1/(x+3)} \cdot \frac{x+3}{x+3} \\ &= \frac{2(x+3)}{(x+3) + \frac{1}{x+3} \cdot (x+3)} \\ &= \frac{2(x+3)}{x+3 + 1} \\ &= \frac{2(x+3)}{x+4}\end{aligned}$$

$$= \frac{2(x+3)}{x+4}$$