

Section 1.3: Polynomials

Def'n: A polynomial is any expression which can be written in the form

$$k_n x^n + k_{n-1} x^{n-1} + k_{n-2} x^{n-2} + \dots + k_2 x^2 + k_1 x + k_0$$

where the natural number n is the degree of the polynomial, and the real numbers $k_0, k_1, \dots, k_n$ are its coefficients.

The constant k_0 is the constant coefficient.

The constant k_n is the leading coefficient.

eg $2x+1$ is a polynomial of degree 1 (a "linear" polynomial)

eg $5-4x^2+x = -4x^2+x+5$
is a polynomial of degree 2
(a "quadratic" polynomial)

eg $x^3 + \frac{1}{4}x^2 - \sqrt{2} = x^3 + \frac{1}{4}x^2 + 0x - \sqrt{2}$
is a polynomial of degree 3
(a "cubic" polynomial)

A constant can be considered a polynomial of degree 0.

A polynomial with exactly two terms is a binomial.

A polynomial with exactly three terms is a trinomial.

We can add or subtract polynomials by adding or subtracting like terms.

$$\begin{aligned} \text{eg } (x^2 - 2x + 4) + (x^2 + 5x - 3) \\ = (x^2 + x^2) + (-2x + 5x) + (4 - 3) \\ = 2x^2 + 3x + 1 \end{aligned}$$

$$\begin{aligned} \text{eg } (2x + 7) - (3x^2 - 8x + 7) \\ = (0x^2 - 3x^2) + (2x - (-8x)) + (7 - 7) \\ = -3x^2 + 10x \end{aligned}$$

To multiply polynomials, we apply the property of distributivity.

$$\begin{aligned} \text{eg } 4x \left(x^2 - \frac{1}{2}x + 4 \right) &= 4x \cdot x^2 + 4x \cdot \left(-\frac{1}{2}x \right) + 4x \cdot 4 \\ &= 4x^3 - 2x^2 + 16x \end{aligned}$$

$$\begin{aligned} \text{eg } (3x + 2)(x - 7) &= 3x(x - 7) + 2(x - 7) \\ &= 3x \cdot x + 3x \cdot (-7) + 2 \cdot x + 2 \cdot (-7) \\ &= 3x^2 - 21x + 2x - 14 \\ &= 3x^2 - 19x - 14 \end{aligned}$$

Some common products include:

$$\begin{aligned}\textcircled{1} (B+C)(B-C) &= B(B-C) + C(B-C) \\ &= B \cdot B + B \cdot (-C) + C \cdot B + C \cdot (-C) \\ &= B^2 - BC + BC - C^2 \\ &= B^2 - C^2\end{aligned}$$

$$\text{eg } (4x-3)(4x+3) = (4x)^2 - 3^2 \\ = 16x^2 - 9$$

$$\begin{aligned}\textcircled{2} (B+C)^2 &= (B+C)(B+C) \\ &= B(B+C) + C(B+C) \\ &= B^2 + BC + BC + C^2 \\ &= B^2 + 2BC + C^2\end{aligned}$$

Observe that $(B+C)^2 \neq B^2 + C^2$

$$\textcircled{3} (B-C)^2 = B^2 - 2BC + C^2$$

$$\text{eg } (x^2-5)^2 = (x^2)^2 - 2x^2 \cdot 5 + 5^2 \\ = x^4 - 10x^2 + 25$$

$$\begin{aligned}\text{eg } (x+2)^3 &= (x+2)^2(x+2) \\ &= (x^2 + 2x \cdot 2 + 2^2)(x+2) \\ &= (x^2 + 4x + 4)(x+2) \\ &= x^2(x+2) + 4x(x+2) + 4(x+2) \\ &= x^3 + 2x^2 + 4x^2 + 8x + 4x + 8 \\ &= x^3 + 6x^2 + 12x + 8\end{aligned}$$

When a polynomial has integer coefficients, it can be factored by rewriting it as a product of lower-degree polynomials with integer coefficients.

eg $x^2 - 9 = (x-3)(x+3)$

eg $x^2 - 2 = (x-\sqrt{2})(x+\sqrt{2})$ but this is not its factored form because not all the coefficients are integers. Instead, we would consider $x^2 - 2$ to already be factored completely.

The simplest way to factor a polynomial is to identify an expression which appears in or is common to every term.

eg $9x^3 - 12x^2 = 3x^2 \cdot 3x - 3x^2 \cdot 4$
 $= 3x^2(3x-4)$

We can sometimes factor polynomials using a special formula:

$$(1) \quad B^2 - C^2 = (B - C)(B + C) \quad (\text{"difference of squares"})$$

$$\text{eg } x^2 - 9 = x^2 - 3^2$$
$$= (x - 3)(x + 3)$$

There is no "sum of squares" formula for $B^2 + C^2$.

$$\text{eg } 81x^4 - 16 = (9x^2)^2 - 4^2$$
$$= (9x^2 - 4)(9x^2 + 4)$$

We cannot factor $9x^2 + 4$ because it is a sum of squares. However,

$$9x^2 - 4 = (3x)^2 - 2^2$$
$$= (3x - 2)(3x + 2)$$

$$\text{Thus } 81x^4 - 16 = (3x - 2)(3x + 2)(9x^2 + 4).$$

$$(2) \quad B^3 - C^3 = (B - C)(B^2 + BC + C^2)$$

$$\text{eg } x^3 - 8 = x^3 - 2^3$$
$$= (x - 2)(x^2 + x \cdot 2 + 2^2)$$
$$= (x - 2)(x^2 + 2x + 4)$$

$$\textcircled{3} \quad B^3 + C^3 = (B+C)(B^2 - BC + C^2)$$

$$\begin{aligned} \text{eg } 64x^3 + 1 &= (4x)^3 + 1^3 \\ &= (4x+1)((4x)^2 - 4x \cdot 1 + 1^2) \\ &= (4x+1)(16x^2 - 4x + 1) \end{aligned}$$

To understand how we might factor a trinomial, consider a polynomial of the form

$$x^2 + Bx + C.$$

This must factor into first-degree polynomials $x+p$ and $x+q$.

$$\begin{aligned} \text{Then } x^2 + Bx + C &= (x+p)(x+q) \\ &= x(x+q) + p(x+q) \\ &= x^2 + qx + px + pq \\ &= x^2 + (p+q)x + pq. \end{aligned}$$

Hence p and q must be chosen so that their product is C and their sum is B .

$$\text{eg Factor } x^2 + 5x + 6.$$

The integers whose product is 6 are:

$1 \cdot 6 = 6$	$1 + 6 = 7$
$-1 \cdot (-6) = 6$	$-1 + (-6) = -7$
$2 \cdot 3 = 6$	$2 + 3 = 5$
$-2 \cdot (-3) = 6$	$-2 + (-3) = -5$

Thus $x^2 + 5x + 6 = (x+2)(x+3)$.

eg Factor $x^2 - x - 12$.

The integers whose product is -12 are:

$$1 \cdot (-12) = -12$$

$$1 + (-12) = -11$$

$$-1 \cdot 12 = -12$$

$$-1 + 12 = 11$$

$$2 \cdot (-6) = -12$$

$$2 + (-6) = -4$$

$$-2 \cdot 6 = -12$$

$$-2 + 6 = 4$$

$$3 \cdot (-4) = -12$$

$$3 + (-4) = -1$$

$$-3 \cdot 4 = -12$$

$$-3 + 4 = 1$$

Thus $x^2 - x - 12 = (x+3)(x-4)$.

Now suppose that the trinomial has the form

$$Ax^2 + Bx + C$$

for $A \neq 1$. Then we have

$$\begin{aligned} Ax^2 + Bx + C &= (kx+p)(lx+q) \\ &= kx(lx+q) + p(lx+q) \\ &= (kl)x^2 + (kq) x + (lp)x + pq \\ &= (kl)x^2 + (kq+lp)x + pq \end{aligned}$$

This means that we still have $pq = C$. Furthermore, we also have $k \cdot l = A$. Finally, $kq + lp = B$.

eg Factor $4x^2 - 8x + 3$.

The integers whose product is 3 are:

$$1 \cdot 3 = 3$$

$$-1 \cdot (-3) = 3$$

Thus either $4x^2 - 8x + 3 = (kx+1)(lx+3)$

or $4x^2 - 8x + 3 = (kx-1)(lx-3)$

The integers whose product is 4 are:

$$1 \cdot 4 = 4$$

$$2 \cdot 2 = 4$$

$$\left[\begin{array}{l} -1 \cdot (-4) = 4 \\ -2 \cdot (-2) = 4 \end{array} \right]$$

Thus we may have

$$4x^2 - 8x + 3 = (x+1)(4x+3) = 4x^2 + 7x + 3$$

$$= (x-1)(4x-3) = 4x^2 - 7x + 3$$

$$= (4x+1)(x+3) = 4x^2 + 13x + 3$$

$$= (4x-1)(x-3) = 4x^2 - 13x + 3$$

$$= (2x+1)(2x+3) = 4x^2 + 8x + 3$$

$$= (2x-1)(2x-3) = 4x^2 - 8x + 3$$

$$\text{Finally, } 4x^2 - 8x + 3 = (2x-1)(2x-3).$$

eg Factor $-4x^2 + 8x - 3$.

We can rewrite this as $-4x^2 + 8x - 3 = -(4x^2 - 8x + 3)$.

$$\text{Hence } -4x^2 + 8x - 3 = -(2x-1)(2x-3).$$

eg Factor $3x^4 - 2x^2 - 1$.

Observe that this polynomial has the same form as $Ax^2 + Bx + C$ with x replaced by x^2 .

Thus the same strategies should apply, resulting in a factored form $(kx^2+p)(lx^2+q)$.

The only integers whose product is -1 are $-1 \cdot 1 = -1$.

Hence we know that $3x^4 - 2x^2 - 1 = (kx^2 - 1)(lx^2 + 1)$.

The only integers that are positive and have a product of 3 are $1 \cdot 3 = 3$.

$$\text{Thus either } 3x^4 - 2x^2 - 1 = (x^2 - 1)(3x^2 + 1) \\ = 3x^4 - 2x^2 - 1$$

$$\text{or } 3x^4 - 2x^2 - 1 = (3x^2 - 1)(x^2 + 1) \\ = 3x^4 + 2x^2 - 1$$

Now we know that

$$3x^4 - 2x^2 - 1 = (x^2 - 1)(3x^2 + 1)$$

$$= (x-1)(x+1)(3x^2+1)$$

We often combine strategies for factoring in order to completely factor a polynomial.

eg Factor $7x^3 + 28x^2 + 21x$.

First we have $7x^3 + 28x^2 + 21x = 7x(x^2 + 4x + 3)$

$$= 7x(x+1)(x+3).$$

A similar approach is factoring by grouping, where we extract common expressions from groups of terms.

eg Factor $x^3 - 3x^2 - 4x + 12$.

We can group the terms of this polynomial as

$$(x^3 - 3x^2) + (-4x + 12)$$

$$= x^2(x-3) - 4(x-3)$$

$$= (x-3)(x^2 - 4)$$

$$= (x-3)(x-2)(x+2).$$