

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

MIDTERM TEST

MATHEMATICS 1001-003

WINTER 2023

SOLUTIONS

- [5] 1. (a) Let $u = x^2$ so $du = 2x dx$ and $\frac{1}{2} du = x dx$. The integral becomes

$$\begin{aligned}\int x \cos(x^2) dx &= \frac{1}{2} \int \cos(u) du \\ &= \frac{1}{2} \sin(u) + C\end{aligned}$$

$$= \frac{1}{2} \sin(x^2) + C.$$

- [6] (b) We use integration by parts with $w = x^2$ so $dw = 2x dx$, and $dv = \cos(x) dx$ so $v = \sin(x)$. Thus

$$\int x^2 \cos(x) dx = x^2 \sin(x) - 2 \int x \sin(x) dx.$$

Now we use integration by parts again, this time with $w = x$ so $dw = dx$, and $dv = \sin(x) dx$ so $v = -\cos(x)$. Now we obtain

$$\begin{aligned}\int x^2 \cos(x) dx &= x^2 \sin(x) - 2 \left[-x \cos(x) + \int \cos(x) dx \right] \\ &= x^2 \sin(x) - 2[-x \cos(x) + \sin(x)] + C\end{aligned}$$

$$= x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C.$$

- [4] (c) We can rewrite the given integral as

$$\begin{aligned}\int \frac{\cos(x) + 1}{\cos^2(x)} dx &= \int \left(\frac{\cos(x)}{\cos^2(x)} + \frac{1}{\cos^2(x)} \right) dx \\ &= \int \sec(x) dx + \int \sec^2(x) dx\end{aligned}$$

$$= \ln|\sec(x) + \tan(x)| + \tan(x) + C.$$

- [25] 2. (a) Let $u = \sinh(3x)$ so $du = 3 \cosh(3x) dx$ and $\frac{1}{3} du = \cosh(3x) dx$. The integral becomes

$$\begin{aligned}\int \frac{\cosh(3x)}{\sinh^2(3x) + 25} dx &= \frac{1}{3} \int \frac{1}{u^2 + 25} du \\ &= \frac{1}{3} \cdot \frac{1}{5} \arctan\left(\frac{u}{5}\right) + C\end{aligned}$$

$$= \frac{1}{15} \arctan\left(\frac{\sinh(3x)}{5}\right) + C.$$

- (b) Let $u = \frac{1}{3}x + 5$ so $du = \frac{1}{3}dx$ and $3du = dx$. Furthermore, $\frac{1}{3}x = u - 5$ so $x = 3u - 15$. Thus the integral becomes

$$\begin{aligned} \int x \left(\frac{1}{3}x + 5 \right)^7 dx &= 3 \int (3u - 15) \cdot u^7 du \\ &= 3 \int (3u^8 - 15u^7) du \\ &= 3 \left[3 \cdot \frac{u^9}{9} - 15 \cdot \frac{u^8}{8} \right] + C \\ &= \left(\frac{1}{3}x + 5 \right)^9 - \frac{45}{8} \left(\frac{1}{3}x + 5 \right)^8 + C. \end{aligned}$$

- (c) Using long division of polynomials, we have

$$\begin{array}{r} 6x^2 - 3x - 1 \\ 3x - 1 \overline{) 18x^3 - 15x^2 + 2} \\ \underline{18x^3 - 6x^2} \\ -9x^2 \\ \underline{-9x^2 + 3x} \\ -3x + 2 \\ \underline{-3x + 1} \\ 1 \end{array}$$

Thus we can write the integral as

$$\begin{aligned} \int \frac{18x^3 - 15x^2 + 2}{3x - 1} dx &= \int \left(6x^2 - 3x - 1 + \frac{1}{3x - 1} \right) dx \\ &= 6 \cdot \frac{x^3}{3} - 3 \cdot \frac{x^2}{2} - x + \frac{1}{3} \ln|3x - 1| + C \\ &= 2x^3 - \frac{3}{2}x^2 - x + \frac{1}{3} \ln|3x - 1| + C. \end{aligned}$$

- (d) Let $u = \ln(x)$ so $du = \frac{1}{x} dx$. The integral becomes

$$\begin{aligned} \int \frac{1}{x \ln(x) \sqrt{\ln^2(x) - 4}} dx &= \int \frac{1}{u \sqrt{u^2 - 4}} du \\ &= \frac{1}{2} \operatorname{arcsec} \left(\frac{u}{2} \right) + C \\ &= \frac{1}{2} \operatorname{arcsec} \left(\frac{\ln(x)}{2} \right) + C. \end{aligned}$$

- (e) Let $u = \sqrt{x}$ so $du = \frac{1}{2\sqrt{x}} dx$ and $dx = 2\sqrt{x} du = 2u du$. The integral becomes

$$\int e^{\sqrt{x}} dx = 2 \int u e^u du.$$

Now we use integration by parts with $w = u$ so $dw = du$, and $dv = e^u du$ so $v = e^u$. This yields

$$\begin{aligned}\int e^{\sqrt{x}} dx &= 2 \left[ue^u - \int e^u du \right] \\ &= 2[ue^u - e^u] + C \\ &= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.\end{aligned}$$

(f) First we complete the square:

$$\begin{aligned}27 + 12x - 4x^2 &= -4 \left[x^2 - 3x - \frac{27}{4} \right] \\ &= -4 \left[\left(x^2 - 3x + \frac{9}{4} \right) - \frac{27}{4} - \frac{9}{4} \right] \\ &= -4 \left[\left(x - \frac{3}{2} \right)^2 - 9 \right] \\ &= 36 - 4 \left(x - \frac{3}{2} \right)^2 \\ &= 36 - (2x - 3)^2.\end{aligned}$$

Now we let $u = 2x - 3$ so $du = 2 dx$ and $\frac{1}{2} du = dx$. The integral becomes

$$\begin{aligned}\int \frac{1}{\sqrt{27 + 12x - 4x^2}} dx &= \int \frac{1}{\sqrt{36 - (2x - 3)^2}} dx \\ &= \frac{1}{2} \int \frac{1}{\sqrt{36 - u^2}} du \\ &= \frac{1}{2} \arcsin \left(\frac{u}{6} \right) + C \\ &= \frac{1}{2} \arcsin \left(\frac{2x - 3}{6} \right) + C.\end{aligned}$$