MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Midterm Test | **MATHEMATICS 1001-002** | Winter 2023

SOLUTIONS

[4] 1. (a) We can rewrite the given integral as

$$\begin{aligned}\int \frac{\cos(x)+1}{\cos^2(x)}\,dx &= \int \left(\frac{\cos(x)}{\cos^2(x)} + \frac{1}{\cos^2(x)}\right) dx \\ &= \int \sec(x)\,dx + \int \sec^2(x)\,dx \\ &= \ln|\sec(x)+\tan(x)| + \tan(x) + C.\end{aligned}$$

[5] (b) Let $u = x^2$ so $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$. The integral becomes

$$\begin{aligned}\int x\cos(x^2)\,dx &= \frac{1}{2}\int \cos(u)\,du \\ &= \frac{1}{2}\sin(u) + C \\ &= \frac{1}{2}\sin(x^2) + C.\end{aligned}$$

[6] (c) We use integration by parts with $w = x^2$ so $dw = 2x\,dx$, and $dv = \cos(x)\,dx$ so $v = \sin(x)$. Thus

$$\int x^2\cos(x)\,dx = x^2\sin(x) - 2\int x\sin(x)\,dx.$$

Now we use integration by parts again, this time with $w = x$ so $dw = dx$, and $dv = \sin(x)\,dx$ so $v = -\cos(x)$. Now we obtain

$$\begin{aligned}\int x^2\cos(x)\,dx &= x^2\sin(x) - 2\left[-x\cos(x) + \int \cos(x)\,dx\right] \\ &= x^2\sin(x) - 2[-x\cos(x) + \sin(x)] + C \\ &= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C.\end{aligned}$$

[25] 2. (a) Let $u = \frac{1}{3}x + 5$ so $du = \frac{1}{3}\,dx$ and $3\,du = dx$. Furthermore, $\frac{1}{3}x = u - 5$ so $x = 3u - 15$. Thus the integral becomes

$$\begin{aligned}\int x\left(\frac{1}{3}x+5\right)^7 dx &= 3\int (3u-15)\cdot u^7\,du \\ &= 3\int (3u^8 - 15u^7)\,du \\ &= 3\left[3\cdot\frac{u^9}{9} - 15\cdot\frac{u^8}{8}\right] + C \\ &= \left(\frac{1}{3}x+5\right)^9 - \frac{45}{8}\left(\frac{1}{3}x+5\right)^8 + C.\end{aligned}$$

(b) Let $u = \sinh(3x)$ so $du = 3\cosh(3x)\,dx$ and $\frac{1}{3}\,du = \cosh(3x)\,dx$. The integral becomes

$$\begin{aligned}\int \frac{\cosh(3x)}{\sinh^2(3x)+25}\,dx &= \frac{1}{3}\int \frac{1}{u^2+25}\,du \\ &= \frac{1}{3}\cdot\frac{1}{5}\arctan\left(\frac{u}{5}\right)+C \\ &= \frac{1}{15}\arctan\left(\frac{\sinh(3x)}{5}\right)+C.\end{aligned}$$

(c) First we complete the square:

$$\begin{aligned}27+12x-4x^2 &= -4\left[x^2-3x-\frac{27}{4}\right] \\ &= -4\left[\left(x^2-3x+\frac{9}{4}\right)-\frac{27}{4}-\frac{9}{4}\right] \\ &= -4\left[\left(x-\frac{3}{2}\right)^2-9\right] \\ &= 36-4\left(x-\frac{3}{2}\right)^2 \\ &= 36-(2x-3)^2.\end{aligned}$$

Now we let $u = 2x-3$ so $du = 2\,dx$ and $\frac{1}{2}\,du = dx$. The integral becomes

$$\begin{aligned}\int \frac{1}{\sqrt{27+12x-4x^2}}\,dx &= \int \frac{1}{\sqrt{36-(2x-3)^2}}\,dx \\ &= \frac{1}{2}\int \frac{1}{\sqrt{36-u^2}}\,du \\ &= \frac{1}{2}\arcsin\left(\frac{u}{6}\right)+C \\ &= \frac{1}{2}\arcsin\left(\frac{2x-3}{6}\right)+C.\end{aligned}$$

(d) Let $u = \sqrt{x}$ so $du = \frac{1}{2\sqrt{x}}\,dx$ and $dx = 2\sqrt{x}\,du = 2u\,du$. The integral becomes

$$\int e^{\sqrt{x}}\,dx = 2\int ue^u\,du.$$

Now we use integration by parts with $w = u$ so $dw = du$, and $dv = e^u\,du$ so $v = e^u$. This yields

$$\begin{aligned}\int e^{\sqrt{x}}\,dx &= 2\left[ue^u-\int e^u\,du\right] \\ &= 2[ue^u-e^u]+C \\ &= 2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+C.\end{aligned}$$

(e) Let $u = \ln(x)$ so $du = \frac{1}{x}\,dx$. The integral becomes

$$\int \frac{1}{x\ln(x)\sqrt{\ln^2(x)-4}}\,dx = \int \frac{1}{u\sqrt{u^2-4}}\,du$$
$$= \frac{1}{2}\operatorname{arcsec}\left(\frac{u}{2}\right) + C$$
$$= \frac{1}{2}\operatorname{arcsec}\left(\frac{\ln(x)}{2}\right) + C.$$

(f) Using long division of polynomials, we have

$$\begin{array}{r|l} & 6x^2 - 3x - 1 \\ \hline 3x-1 & 18x^3 - 15x^2 \qquad + 2 \\ & \underline{18x^3 - 6x^2} \\ & \quad -9x^2 \qquad + 2 \\ & \quad \underline{-9x^2 + 3x} \\ & \qquad -3x + 2 \\ & \qquad \underline{-3x + 1} \\ & \qquad\qquad 1 \end{array}$$

Thus we can write the integral as

$$\int \frac{18x^3 - 15x^2 + 2}{3x-1}\,dx = \int \left(6x^2 - 3x - 1 + \frac{1}{3x-1}\right)dx$$
$$= 6\cdot\frac{x^3}{3} - 3\cdot\frac{x^2}{2} - x + \frac{1}{3}\ln|3x-1| + C$$
$$= 2x^3 - \frac{3}{2}x^2 - x + \frac{1}{3}\ln|3x-1| + C.$$