MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 3.4 **Math 1001 Worksheet** WINTER 2023

SOLUTIONS

1. (a) We have

$$\begin{aligned}\int_2^\infty \frac{1}{\sqrt{4x+1}}\,dx &= \lim_{t\to\infty}\int_2^\infty \frac{1}{\sqrt{4x+1}}\,dx \\ &= \lim_{t\to\infty}\left[\frac{1}{2}\sqrt{4x+1}\right]_2^t \\ &= \lim_{t\to\infty}\left(\frac{1}{2}\sqrt{4t+1}-\frac{3}{2}\right) \\ &= \infty.\end{aligned}$$

So the integral diverges.

(b) We write

$$\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{t\to 4^-}\int_0^t \frac{1}{\sqrt{4-x}}\,dx = \lim_{t\to 4^-}\left[-2\sqrt{4-x}\right]_0^t = \lim_{t\to 4^-}(-2\sqrt{4-t}+4) = 4.$$

So the integral converges.

(c) We have

$$\int_{-\infty}^0 e^{3x}\,dx = \lim_{t\to-\infty}\int_t^0 e^{3x}\,dx = \lim_{t\to-\infty}\left[\frac{1}{3}e^{3x}\right]_t^0 = \lim_{t\to-\infty}\left(\frac{1}{3}-\frac{1}{3}e^{3t}\right) = \frac{1}{3}-0 = \frac{1}{3}.$$

So the integral converges.

(d) We write

$$\begin{aligned}\int_1^\infty \frac{1}{(x+3)^{\frac{3}{2}}}\,dx &= \lim_{t\to\infty}\int_1^t (x+3)^{-\frac{3}{2}}\,dx \\ &= \lim_{t\to\infty}\left[\frac{-2}{\sqrt{x+3}}\right]_1^t \\ &= \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t+3}}+1\right) \\ &= -0+1 \\ &= 1.\end{aligned}$$

So the integral converges.

(e) First we write

$$\int_0^\infty \frac{x}{x^4+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2)^2+1}\,dx.$$

Let $u = x^2$ so $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$. When $x = 0$, $u = 0$. When $x = t$, $u = t^2$. Thus we have

$$\begin{aligned}\int_0^\infty \frac{x}{x^4+1}\,dx &= \frac{1}{2}\lim_{t\to\infty}\int_0^{t^2} \frac{1}{u^2+1}\,du \\ &= \frac{1}{2}\lim_{t\to\infty}\Big[\arctan(u)\Big]_0^{t^2} \\ &= \frac{1}{2}\lim_{t\to\infty}[\arctan(t^2) - 0] \\ &= \frac{1}{2}\left(\frac{\pi}{2}\right) \\ &= \frac{\pi}{4}.\end{aligned}$$

So the integral converges.

(f) We write

$$\int_{-\infty}^0 \frac{e^x}{1+e^x}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{e^x}{1+e^x}\,dx.$$

Let $u = 1 + e^x$ so $du = e^x\,dx$. When $x = t$, $u = 1 + e^t$. When $x = 0$, $u = 2$. Thus we obtain

$$\begin{aligned}\int_{-\infty}^0 \frac{e^x}{1+e^x}\,dx &= \lim_{t\to-\infty}\int_{1+e^t}^2 \frac{1}{u}\,du \\ &= \lim_{t\to-\infty}\Big[\ln|u|\Big]_{1+e^t}^2 \\ &= \lim_{t\to-\infty}[\ln(2) - \ln(1+e^t)] \\ &= \ln(2) - \ln(1+0) \\ &= \ln(2).\end{aligned}$$

So the integral converges.

(g) We have

$$\begin{aligned}\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx &= \lim_{t\to 3^-}\int_0^t \frac{1}{\sqrt{9-x^2}}\,dx\\ &= \lim_{t\to 3^-}\left[\arcsin\left(\frac{x}{3}\right)\right]_0^t\\ &= \lim_{t\to 3^-}\left[\arcsin\left(\frac{t}{3}\right)-\arcsin(0)\right]\\ &= \frac{\pi}{2}-0\\ &= \frac{\pi}{2}.\end{aligned}$$

So the integral converges.

(h) We write

$$\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx = \lim_{t\to 3^-}\int_0^t \frac{x}{\sqrt{9-x^2}}\,dx.$$

Let $u = 9-x^2$ so $du = -2x\,dx$ and $-\frac{1}{2}\,dx = x\,dx$. When $x = 0$, $u = 9$. When $x = t$, $u = 9-t^2$. This yields

$$\begin{aligned}\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx &= -\frac{1}{2}\lim_{t\to 3^-}\int_9^{9-t^2} \frac{1}{\sqrt{u}}\,du\\ &= -\frac{1}{2}\lim_{t\to 3^-}\left[2\sqrt{u}\right]_9^{9-t^2}\\ &= -\lim_{t\to 3^-}\left[\sqrt{9-t^2}-3\right]\\ &= -(0-3)\\ &= 3.\end{aligned}$$

So the integral converges.

(i) We have

$$\int_e^\infty \frac{1}{x\ln^2(x)}\,dx = \lim_{t\to\infty}\int_e^t \frac{1}{x\ln^2(x)}\,dx.$$

Let $u = \ln(x)$ so $du = \frac{1}{x}\,dx$. When $x = e$, $u = 1$. When $x = t$, $u = \ln(t)$. Thus we have

$$\begin{aligned}
\int_e^\infty \frac{1}{x\ln^2(x)}\,dx &= \lim_{t\to\infty} \int_1^{\ln(t)} \frac{1}{u^2}\,du \\
&= \lim_{t\to\infty} \left[-\frac{1}{u}\right]_1^{\ln(t)} \\
&= \lim_{t\to\infty} \left(-\frac{1}{\ln(t)} + 1\right) \\
&= -0 + 1 \\
&= 1.
\end{aligned}$$

So the integral converges.

(j) We write

$$\begin{aligned}
\int_{-\infty}^{\frac{3}{2}} \frac{1}{4x^2+9}\,dx &= \lim_{t\to-\infty} \int_t^{\frac{3}{2}} \frac{1}{(2x)^2+9}\,dx \\
&= \lim_{t\to-\infty} \left[\frac{1}{6}\arctan\left(\frac{2x}{3}\right)\right]_t^{\frac{3}{2}} \\
&= \lim_{t\to-\infty} \left[\frac{1}{6}\left(\frac{\pi}{4}\right) - \frac{1}{6}\arctan\left(\frac{2t}{3}\right)\right] \\
&= \frac{\pi}{24} - \frac{1}{6}\left(-\frac{\pi}{2}\right) \\
&= \frac{\pi}{24} + \frac{\pi}{12} \\
&= \frac{\pi}{8}.
\end{aligned}$$

So the integral converges.

(k) Using integration by parts with $w = x$ and $dv = e^{-x}\,dx$, we get

$$\begin{aligned}
\int_0^\infty xe^{-x}\,dx &= \lim_{t\to\infty} \int_0^t xe^{-x}\,dx \\
&= \lim_{t\to\infty} \left(\left[-xe^{-x}\right]_0^t + \int_0^t e^{-x}\,dx \right) \\
&= \lim_{t\to\infty} \left[-xe^{-x} - e^{-x}\right]_0^t \\
&= \lim_{t\to\infty} \left(-te^{-t} - e^{-t} + 1\right) \\
&= \lim_{t\to\infty} \frac{-t}{e^t} - \lim_{t\to\infty} e^{-t} + \lim_{t\to\infty} 1 \\
&= \lim_{t\to\infty} \frac{-t}{e^t} - 0 + 1 \\
&\overset{\text{H}}{=} \lim_{t\to\infty} \frac{-1}{e^t} + 1 \\
&= 0 + 1 \\
&= 1.
\end{aligned}$$

So the integral converges.

(ℓ) Using integration by parts with $w = \ln(x)$ and $dv = \frac{1}{x\sqrt{x}}\,dx$, we obtain

$$\begin{aligned}
\int_1^\infty \frac{\ln(x)}{x\sqrt{x}}\,dx &= \lim_{t\to\infty} \int_1^t \frac{\ln(x)}{x\sqrt{x}}\,dx \\
&= \lim_{t\to\infty} \left(\left[-\frac{2\ln(x)}{\sqrt{x}} \right]_1^t + \int_1^t \frac{2}{x\sqrt{x}}\,dx \right) \\
&= \lim_{t\to\infty} \left[-\frac{2\ln(x)}{\sqrt{x}} - \frac{4}{\sqrt{x}} \right]_1^t \\
&= \lim_{t\to\infty} \left(-\frac{2\ln(t)}{\sqrt{t}} - \frac{4}{\sqrt{t}} + 4 \right) \\
&= \lim_{t\to\infty} \frac{-2\ln(t)}{\sqrt{t}} - \lim_{t\to\infty} \frac{4}{\sqrt{t}} + \lim_{t\to\infty} 4 \\
&= \lim_{t\to\infty} \frac{-2\ln(t)}{\sqrt{t}} - 0 + 4 \\
&\overset{\text{H}}{=} \lim_{t\to\infty} \frac{-\frac{2}{t}}{\frac{1}{2\sqrt{t}}} + 4 \\
&= \lim_{t\to\infty} \frac{-4}{\sqrt{t}} + 4 \\
&= -0 + 4 \\
&= 4.
\end{aligned}$$

So the integral converges.