MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 3.3 | **Math 1001 Worksheet** | WINTER 2022

SOLUTIONS

1. (a) We let $x = 6\sec(\theta)$ so $dx = 6\sec(\theta)\tan(\theta)\,d\theta$ and $\sqrt{x^2 - 36} = 6\tan(\theta)$. The integral becomes

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-36}}\,dx &= \int \frac{6\sec(\theta)\tan(\theta)}{6\tan(\theta)}\,d\theta\\ &= \int \sec(\theta)\,d\theta\\ &= \ln|\sec(\theta)+\tan(\theta)| + C\\ &= \ln\left|\frac{x}{6} + \frac{\sqrt{x^2-36}}{6}\right| + C.\end{aligned}$$

(b) We let $x = \sqrt{2}\sin(\theta)$, so $dx = \sqrt{2}\cos(\theta)\,d\theta$, $\sqrt{2-x^2} = \sqrt{2}\cos(\theta)$ and so $(2-x^2)^{\frac{3}{2}} = 2\sqrt{2}\cos^3(\theta)$. The integral becomes

$$\begin{aligned}\int \frac{1}{(2-x^2)^{\frac{3}{2}}}\,dx &= \int \frac{\sqrt{2}\cos(\theta)}{2\sqrt{2}\cos^3(\theta)}\,d\theta\\ &= \frac{1}{2}\int \sec^2(\theta)\,d\theta\\ &= \frac{1}{2}\tan(\theta) + C.\end{aligned}$$

Now note that

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{1}{\sqrt{2}}x}{\frac{1}{\sqrt{2}}\sqrt{2-x^2}} = \frac{x}{\sqrt{2-x^2}}.$$

Thus

$$\int \frac{1}{(2-x^2)^{\frac{3}{2}}}\,dx = \frac{x}{2\sqrt{2-x^2}} + C.$$

(c) We let $x = 4\sin(\theta)$ so $dx = 4\cos(\theta)\,d\theta$ and $\sqrt{16-x^2} = 4\cos(\theta)$. Also note that $x = 0$ implies $\sin(\theta) = 0$ and so $\theta = 0$, while $x = 2$ implies $\sin(\theta) = \frac{1}{2}$ and so $\theta = \frac{\pi}{6}$. The integral therefore becomes

$$\int_0^2 \sqrt{16-x^2}\,dx = \int_0^{\frac{\pi}{6}} [4\cos(\theta)]\cdot 4\cos(\theta)\,d\theta$$

$$= 16\int_0^{\frac{\pi}{6}} \cos^2(\theta)\,d\theta$$

$$= 16\int_0^{\frac{\pi}{6}} \left[\frac{1+\cos(2\theta)}{2}\right]\,d\theta$$

$$= 8\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{6}}$$

$$= 8\left[\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right]$$

$$= \frac{4\pi}{3} + 2\sqrt{3}.$$

2. Note that

$$\sqrt{e^{6x}-4} = \sqrt{(e^{3x})^2-4}.$$

We let $u = e^{3x}$, so $du = 3e^{3x}\,dx = 3u\,dx$ and thus $\frac{1}{3u}\,du = dx$. Now we have

$$\int \sqrt{e^{6x}-4}\,dx = \frac{1}{3}\int \frac{\sqrt{u^2-4}}{u}\,du.$$

Next we let $u = 2\sec(\theta)$ so $du = 2\sec(\theta)\tan(\theta)\,d\theta$ and $\sqrt{u^2-4} = 2\tan(\theta)$. The integral becomes

$$\int \sqrt{e^{6x}-4}\,dx = \frac{1}{3}\int \frac{2\tan(\theta)}{2\sec(\theta)}\cdot 2\sec(\theta)\tan(\theta)\,d\theta$$

$$= \frac{2}{3}\int \tan^2(\theta)\,d\theta$$

$$= \frac{2}{3}\int [\sec^2(\theta)-1]\,d\theta$$

$$= \frac{2}{3}[\tan(\theta)-\theta] + C$$

$$= \frac{2}{3}\left[\frac{1}{2}\sqrt{u^2-4} - \operatorname{arcsec}\left(\frac{u}{2}\right)\right] + C$$

$$= \frac{2}{3}\left[\frac{1}{2}\sqrt{e^{6x}-4} - \operatorname{arcsec}\left(\frac{e^{3x}}{2}\right)\right] + C$$

$$= \frac{1}{3}\sqrt{e^{6x}-4} - \frac{2}{3}\operatorname{arcsec}\left(\frac{e^{3x}}{2}\right) + C.$$

3. We complete the square:

$$\begin{aligned}4x - x^2 &= -[x^2 - 4x] \\ &= -[(x^2 - 4x + 4) - 4] \\ &= -[(x-2)^2 - 4] \\ &= 4 - (x-2)^2.\end{aligned}$$

Thus we can rewrite the integral as

$$\int \frac{x^2}{\sqrt{4x - x^2}}\,dx = \int \frac{x^2}{\sqrt{4-(x-2)^2}}\,dx.$$

Now let $u = x - 2$ so $du = dx$ and $x = u + 2$. Then we have

$$\int \frac{x^2}{\sqrt{4-(x-2)^2}}\,dx = \int \frac{(u+2)^2}{\sqrt{4-u^2}}\,du = \int \frac{u^2+4u+4}{\sqrt{4-u^2}}\,du.$$

Now we let $u = 2\sin(\theta)$ so $du = 2\cos(\theta)\,d\theta$ and $\sqrt{4-u^2} = 2\cos(\theta)$. The integral becomes

$$\begin{aligned}&\int \frac{x^2}{\sqrt{4x-x^2}}\,dx \\ &= \int \frac{4 + 8\sin(\theta) + 4\sin^2(\theta)}{2\cos(\theta)} \cdot 2\cos(\theta)\,d\theta \\ &= \int [4 + 8\sin(\theta) + 4\sin^2(\theta)]\,d\theta \\ &= \int [4 + 8\sin(\theta) + 2(1 - \cos(2\theta))]\,d\theta \\ &= \int [6 + 8\sin(\theta) - 2\cos(2\theta)]\,d\theta \\ &= 6\theta - 8\cos(\theta) - \sin(2\theta) + C \\ &= 6\theta - 8\cos(\theta) - 2\sin(\theta)\cos(\theta) + C \\ &= 6\arcsin\left(\frac{u}{2}\right) - 8\cdot\frac{\sqrt{4-u^2}}{2} - 2\cdot\frac{u}{2}\cdot\frac{\sqrt{4-u^2}}{2} + C \\ &= 6\arcsin\left(\frac{u}{2}\right) - 4\sqrt{4-u^2} - \frac{1}{2}u\sqrt{4-u^2} + C \\ &= 6\arcsin\left(\frac{x-2}{2}\right) - 4\sqrt{4-(x-2)^2} - \frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + C \\ &= 6\arcsin\left(\frac{x-2}{2}\right) - 3\sqrt{4-(x-2)^2} - \frac{1}{2}x\sqrt{4-(x-2)^2} + C.\end{aligned}$$

4. (a) We use u-substitution, with $u = x^2 + 4$ so $\frac{1}{2}\,du = x\,dx$. Then

$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$
$$= \frac{1}{2}\left[2\sqrt{u}\right] + C$$
$$= \sqrt{x^2+4} + C.$$

(b) We use trigonometric substitution, with $x = 2\sin(\theta)$. Then $dx = 2\cos(\theta)\,d\theta$ and $\sqrt{4-x^2} = 2\cos(\theta)$. The integral becomes

$$\int \frac{\sqrt{4-x^2}}{x}\,dx = \int \frac{2\cos(\theta)}{2\sin(\theta)} \cdot 2\cos(\theta)\,d\theta$$
$$= 2\int \frac{\cos^2(\theta)}{\sin(\theta)}\,d\theta$$
$$= 2\int \frac{1-\sin^2(\theta)}{\sin(\theta)}\,d\theta$$
$$= 2\int \left[\csc(\theta) - \sin(\theta)\right]d\theta$$
$$= -2\ln|\csc(\theta) + \cot(\theta)| + 2\cos(\theta) + C.$$

We already know that $\sin(\theta) = \frac{1}{2}x$, so $\csc(\theta) = \frac{2}{x}$. Furthermore, $\cos(\theta) = \frac{1}{2}\sqrt{4-x^2}$. Finally,

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\frac{1}{2}\sqrt{4-x^2}}{\frac{1}{2}x} = \frac{\sqrt{4-x^2}}{x}.$$

Hence

$$\int \frac{\sqrt{4-x^2}}{x}\,dx = -2\ln\left|\frac{2}{x} + \frac{\sqrt{4-x^2}}{x}\right| + 2\left(\frac{1}{2}\sqrt{4-x^2}\right) + C$$
$$= -2\ln\left|\frac{\sqrt{4-x^2}+2}{x}\right| + \sqrt{4-x^2} + C.$$

(c) We complete the square:

$$4x^2 - 12x + 13 = 4\left[x^2 - 3x + \frac{13}{4}\right]$$
$$= 4\left[\left(x^2 - 3x + \frac{9}{4}\right) + 1\right]$$
$$= 4\left(x - \frac{3}{2}\right)^2 + 4$$
$$= (2x-3)^2 + 4.$$

Thus

$$\int \frac{1}{4x^2 - 12x + 13}\,dx = \int \frac{1}{(2x-3)^2 + 4}\,dx.$$

Let $u = 2x - 3$ so $\frac{1}{2}\,du = dx$. The integral becomes

$$\frac{1}{2}\int \frac{1}{u^2+4}\,du = \frac{1}{2}\cdot\frac{1}{2}\arctan\left(\frac{u}{2}\right) + C$$
$$= \frac{1}{4}\arctan\left(\frac{2x-3}{2}\right) + C.$$

(d) We can factor the denominator:

$$4x^2 - 4x - 3 = (2x+1)(2x-3).$$

Thus an appropriate partial fraction decomposition is

$$\frac{1}{4x^2-4x-3} = \frac{A}{2x+1} + \frac{B}{2x-3}$$
$$1 = A(2x-3) + B(2x+1).$$

When $x = -\frac{1}{2}$, we get $1 = -4A$ so $A = -\frac{1}{4}$. When $x = \frac{3}{2}$, we get $1 = 4B$ so $B = \frac{1}{4}$. Thus

$$\frac{1}{4x^2-4x-3}\,dx = \int \left(\frac{-\frac{1}{4}}{2x+1} + \frac{\frac{1}{4}}{2x-3}\right)dx$$
$$= -\frac{1}{4}\cdot\frac{1}{2}\ln|2x+1| + \frac{1}{4}\cdot\frac{1}{2}\ln|2x-3| + C$$
$$= \frac{1}{8}\ln|2x-3| - \frac{1}{8}\ln|2x+1| + C.$$

(e) We use u-substitution, with $u = \ln(x)$ so $du = \frac{1}{x}\,dx$. The integral becomes

$$\int \frac{1}{x[\ln(x)]^2}\,dx = \int \frac{1}{u^2}\,du$$
$$= -\frac{1}{u} + C$$
$$= -\frac{1}{\ln(x)} + C.$$

(f) We use integration by parts. Let $w = [\ln(x)]^2$ so $dw = \dfrac{2\ln(x)}{x}\,dx$. Let $dv = x\,dx$ so $v = \frac{1}{2}x^2$. Then

$$\int x[\ln(x)]^2\,dx = \frac{1}{2}x^2[\ln(x)]^2 - \int x\ln(x)\,dx.$$

We need to use integration by parts a second time. Let $w = \ln(x)$ so $dw = \frac{1}{x}\,dx$, and let $dv = x\,dx$ so $v = \frac{1}{2}x^2$. Now

$$\begin{aligned}\int x[\ln(x)]^2\,dx &= \frac{1}{2}x^2[\ln(x)]^2 - \left[\frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,dx\right]\\ &= \frac{1}{2}x^2[\ln(x)]^2 - \frac{1}{2}x^2\ln(x) + \frac{1}{4}x^2 + C.\end{aligned}$$

(g) We use a trig integration strategy. Since the powers of $\sin(x)$ and $\cos(x)$ are both odd, we can two one of two approaches.

First, we could factor out a $\cos(x)$ and then use the identity

$$\cos^2(x) = 1 - \sin^2(x)$$

to obtain

$$\begin{aligned}\int \sin^3(x)\cos^3(x)\,dx &= \int \sin^3(x)\cos^2(x)\cdot\cos(x)\,dx\\ &= \int \sin^3(x)[1-\sin^2(x)]\cos(x)\,dx.\end{aligned}$$

Now let $u = \sin(x)$ so $du = \cos(x)\,dx$. The integral becomes

$$\begin{aligned}\int \sin^3(x)\cos^3(x)\,dx &= \int u^3[1-u^2]\,du\\ &= \int [u^3 - u^5]\,du\\ &= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C\\ &= \frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) + C.\end{aligned}$$

Alternatively, we could factor out a $\sin(x)$ and then use the identity

$$\sin^2(x) = 1 - \cos^2(x)$$

to obtain

$$\begin{aligned}\int \sin^3(x)\cos^3(x)\,dx &= \int \sin^2(x)\cos^3(x)\cdot\sin(x)\,dx\\ &= \int [1-\cos^2(x)]\cos^3(x)\cdot\sin(x)\,dx.\end{aligned}$$

Now let $u = \cos(x)$ so $du = -\sin(x)\,dx$ and $-\,du = \sin(x)\,dx$. The integral becomes

$$\begin{aligned}\int \sin^3(x)\cos^3(x)\,dx &= -\int [1-u^2]u^3\,du \\ &= -\int [u^3 - u^5]\,du \\ &= \frac{1}{6}u^6 - \frac{1}{4}u^4 + C \\ &= \frac{1}{6}\cos^6(x) - \frac{1}{4}\cos^4(x) + C.\end{aligned}$$

(h) Here we simply have to realise that

$$\sin^3(x)\csc^3(x) = \sin^3(x)\cdot\frac{1}{\sin^3(x)} = 1,$$

so

$$\int \sin^3(x)\csc^3(x)\,dx = \int dx = x + C.$$

(i) This is trickier. First note that this is not an elementary integral, and we can rule out most techniques: u-substitution, trig substitution, partial fractions, trig integrals. That pretty much means that we must be able to use integration by parts. However, as written, parts doesn't get us anywhere: there's no choice of w and dv that will provide a manageable integral. So, instead, let's rewrite the integral as

$$\int x[\sec^2(x) - 1]\,dx = \int x\sec^2(x)\,dx - \int x\,dx = \int x\sec^2(x)\,dx - \frac{1}{2}x^2.$$

For the remaining integral, parts is a viable option. Let $w = x$ so $dw = dx$. Let $dv = \sec^2(x)\,dx$ so $v = \tan(x)$. Then

$$\begin{aligned}\int x\sec^2(x)\,dx &= x\tan(x) - \int \tan(x)\,dx \\ &= x\tan(x) + \ln|\cos(x)| + C.\end{aligned}$$

Finally, then,

$$\int x\tan^2(x)\,dx = x\tan(x) + \ln|\cos(x)| - \frac{1}{2}x^2 + C.$$