MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 2 | **MATHEMATICS 1001** | Winter 2023

SOLUTIONS

[6] 1. (a) We can write

$$\int \left[x^3 \sin(x^4) - \frac{\cot\left(\frac{1}{x^4}\right)}{x^5} \right] dx = \int x^3 \sin(x^4)\, dx - \int \frac{\cot\left(\frac{1}{x^4}\right)}{x^5}\, dx$$

and evaluate the two integrals separately. For the first integral, let $u = x^4$ so $du = 4x^3\, dx$ and $\frac{1}{4}\, du = x^3\, dx$. Then

$$\int x^3 \sin(x^4)\, dx = \frac{1}{4} \int \sin(u)\, du = -\frac{1}{4}\cos(u) + C = -\frac{1}{4}\cos(x^4) + C.$$

For the second integral, let $z = x^{-4}$ so $dz = -4x^{-5}\, dx$ and $-\frac{1}{4}\, dz = x^{-5}\, dx$. Then

$$\begin{aligned} \int \frac{\cot\left(\frac{1}{x^4}\right)}{x^5}\, dx &= -\frac{1}{4} \int \cot(z)\, dz = -\frac{1}{4} \ln|\sin(z)| + C \\ &= -\frac{1}{4} \ln|\sin(x^{-4})| + C \\ &= -\frac{1}{4} \ln\left|\sin\left(\frac{1}{x^4}\right)\right| + C. \end{aligned}$$

Thus

$$\begin{aligned} \int \left[x^3 \sin(x^4) - \frac{\cot\left(\frac{1}{x^4}\right)}{x^5} \right] dx &= -\frac{1}{4}\cos(x^4) - \left[-\frac{1}{4} \ln\left|\sin\left(\frac{1}{x^4}\right)\right| \right] + C \\ &= \frac{1}{4} \ln\left|\sin\left(\frac{1}{x^4}\right)\right| - \frac{1}{4}\cos(x^4) + C. \end{aligned}$$

[5] (b) Let $u = x^2 - 3$ so $du = 2x\, dx$ and $\frac{1}{2}\, du = x\, dx$. Then we can write

$$\int x^7 \sqrt{x^2 - 3}\, dx = \int x^6 \sqrt{x^2 - 3} \cdot x\, dx = \frac{1}{2} \int x^6 \sqrt{u}\, du.$$

Now we need to write x^6 in terms of u, so we observe that

$$x^2 = u + 3 \quad \Longrightarrow \quad x^6 = (u + 3)^3.$$

Thus

$$\begin{aligned}
\int x^7\sqrt{x^2-3}\,dx &= \frac{1}{2}\int (u+3)^3\sqrt{u}\,du \\
&= \frac{1}{2}\int (u^3+9u^2+27u+27)\sqrt{u}\,du \\
&= \frac{1}{2}\int (u^{\frac{7}{2}}+9u^{\frac{5}{2}}+27u^{\frac{3}{2}}+27u^{\frac{1}{2}})\,du \\
&= \frac{1}{2}\left[\frac{2}{9}u^{\frac{9}{2}}+9\cdot\frac{2}{7}u^{\frac{7}{2}}+27\cdot\frac{2}{5}u^{\frac{5}{2}}+27\cdot\frac{2}{3}u^{\frac{3}{2}}\right]+C \\
&= \frac{1}{9}(x^2-3)^{\frac{9}{2}}+\frac{9}{7}(x^2-3)^{\frac{7}{2}}+\frac{27}{5}(x^2-3)^{\frac{5}{2}}+9(x^2-3)^{\frac{3}{2}}+C.
\end{aligned}$$

[3] (c) Observe that

$$\int \frac{20z^3+5z}{6z^4+3z^2+11}\,dz = 5\int \frac{4z^3+z}{6z^4+3z^2+11}\,dz.$$

Let $u=6z^4+3z^2+11$ so $du=(24z^3+6z)\,dz=6(4z^3+z)\,dz$ and $\frac{1}{6}\,dz=(4z^3+z)\,dz$. The integral becomes

$$\begin{aligned}
\int \frac{20z^3+5z}{6z^4+3z^2+11}\,dz &= 5\cdot\frac{1}{6}\int\frac{1}{u}\,du \\
&= \frac{5}{6}\ln|u|+C \\
&= \frac{5}{6}\ln(6z^4+3z^2+11)+C,
\end{aligned}$$

where we can drop the absolute value in the argument of the logarithm because $6z^4+3z^2+11$ must be positive for all z.

[6] (d) By long division, we have:

$$\begin{array}{r l}
 & 3x^3 - 2x - 3 \\
x^2+4\,\big) & 3x^5 + 10x^3 - 3x^2 \qquad - 12 \\
 & \underline{3x^5 + 12x^3} \\
 & \quad -2x^3 - 3x^2 \qquad - 12 \\
 & \quad \underline{-2x^3 \qquad - 8x} \\
 & \qquad -3x^2 + 8x - 12 \\
 & \qquad \underline{-3x^2 \qquad - 12} \\
 & \qquad\qquad 8x
\end{array}$$

Thus we can write

$$\frac{3x^5+10x^3-3x^2-12}{x^2+4} = 3x^3-2x-3+\frac{8x}{x^2+4}.$$

Hence

$$\int \frac{3x^5 + 10x^3 - 3x^2 - 12}{x^2+4}\,dx = \int (3x^3 - 2x - 3)\,dx + 8\int \frac{x}{x^2+4}\,dx$$
$$= \frac{3}{4}x^4 - x^2 - 3x + 8\int \frac{x}{x^2+4}\,dx.$$

For the remaining integral, we can let $u = x^2 + 4$ so $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$. Thus

$$\int \frac{3x^5 + 10x^3 - 3x^2 - 12}{x^2+4}\,dx = \frac{3}{4}x^4 - x^2 - 3x + 8\cdot\frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{3}{4}x^4 - x^2 - 3x + 4\ln|u| + C$$
$$= \frac{3}{4}x^4 - x^2 - 3x + 4\ln(x^2+4) + C,$$

where we can drop the absolute value in the argument of the logarithm because $x^2 + 4$ must be positive for all x.