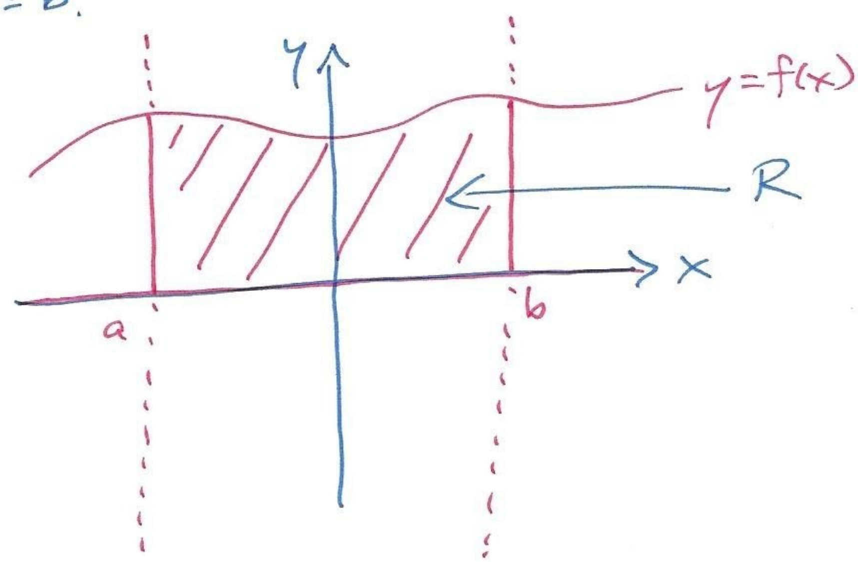
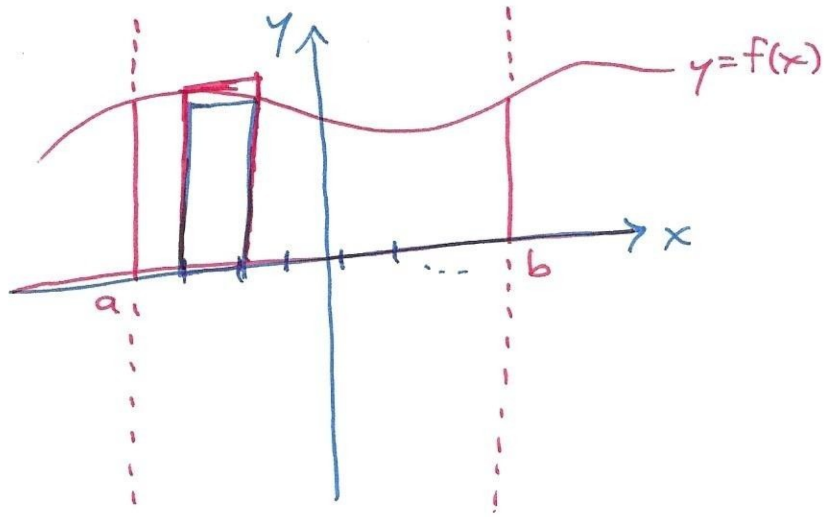
Section 2.1: Area Under a Curve

We know how to find the area of elementary shapes like rectangles, triangles and circles. But how can we find the area of a region that is <u>not</u> simply a combination of these shapes?

Specifically, consider a region R which is bounded above by a curve $y = f(x)$, bounded below by the x-axis, bounded to the left by $x = a$, and bounded to the right by $x = b$.



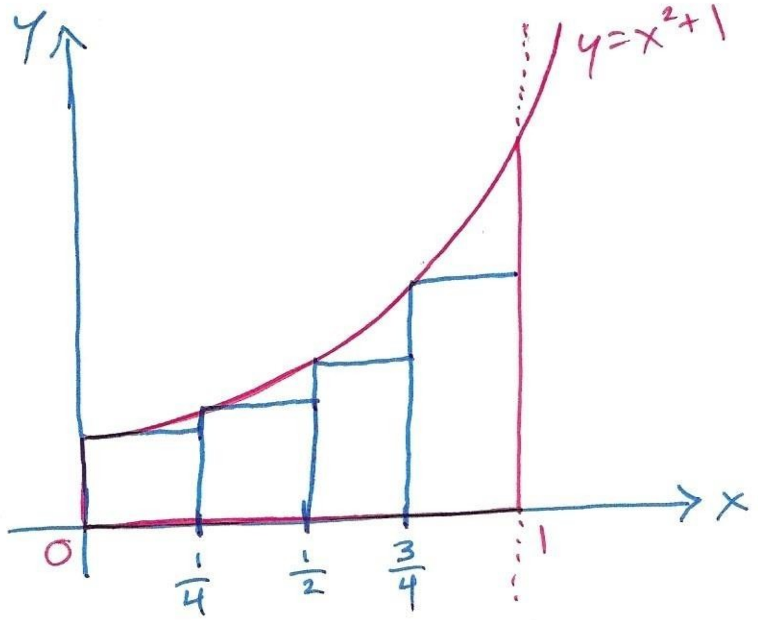
We want to determine the exact value of the area A of the region R.



In order to find an approximation of the area A, we will approximate the region R with a number n of rectangles. Then we could compute the area of each rectangle, and estimate A by taking their sum.

We will divide the interval $[a,b]$ or $a \leq x \leq b$ into n subintervals, and use those subintervals as one side of each rectangle. We call this a partition of $[a,b]$. If each subinterval has the same width Δx then we call it a regular partition.

eg Estimate the area bounded above by $f(x)=x^2+1$, below by the x-axis, to the left by $x=0$, and to the right by $x=1$. Do so with a regular partition into 4 subintervals.



We want to divide $[0,1]$ into 4 equal subintervals. Thus they will have width

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

which will also be the width of each rectangle.

Thus the first rectangle will be drawn on the subinterval $[0, 1/4]$.

" second rectangle " $[1/4, 1/2]$.

" third rectangle " $[1/2, 3/4]$.

" fourth rectangle " $[3/4, 1]$.

For the height of each rectangle, we will choose the minimum value of $f(x)$ on the corresponding subinterval. These are called <u>inscribed</u> rectangles.

For the first rectangle, its height will be $f(0)=1$, so its area is $A_1 = \frac{1}{4} \cdot 1 = \frac{1}{4}$.

For the second rectangle, its height will be $f(\frac{1}{4}) = \frac{17}{16}$, so its area is $A_2 = \frac{1}{4} \cdot \frac{17}{16} = \frac{17}{64}$.

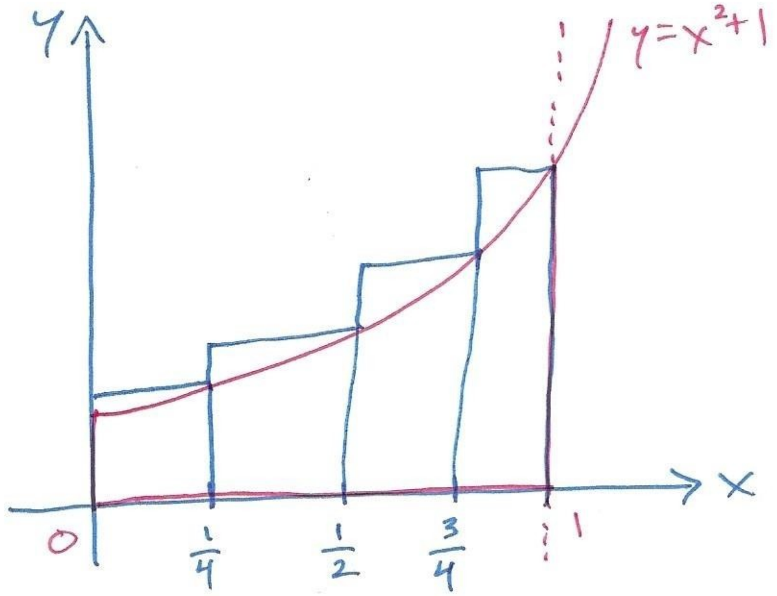
For the third rectangle, its height will be $f(\frac{1}{2}) = \frac{5}{4}$, so its area is $A_3 = \frac{1}{4} \cdot \frac{5}{4} = \frac{5}{16}$.

For the fourth rectangle, its height will $f(\frac{3}{4}) = \frac{25}{16}$, so its area is $A_4 = \frac{1}{4} \cdot \frac{25}{16} = \frac{25}{64}$.

Then we can estimate the true area A to be

$$A \approx A_1 + A_2 + A_3 + A_4 = \frac{39}{32} \approx 1.22.$$

This is the smallest reasonable estimate of A, and is therefore called the <u>lower sum</u>.



Instead, we could try using the maximum value of $f(x)$ on the corresponding interval to give the height of each rectangle. These are called <u>circumscribed</u> rectangles.

For the first rectangle, its height will $f(\frac{1}{4}) = \frac{17}{16}$, so its area is $B_1 = \frac{1}{4} \cdot \frac{17}{16} = \frac{17}{64}$.

Likewise, the other rectangles will have heights of $f(\frac{1}{2}) = \frac{5}{4}$, $f(\frac{3}{4}) = \frac{25}{16}$ and $f(1) = 2$ so their areas will be

$$B_2 = \frac{1}{4} \cdot \frac{5}{4} = \frac{5}{16}, \quad B_3 = \frac{1}{4} \cdot \frac{25}{16} = \frac{25}{64}, \quad B_4 = \frac{1}{4} \cdot 2 = \frac{1}{2}.$$

Our new estimate of A is

$$A \approx B_1 + B_2 + B_3 + B_4 = \frac{47}{32} \approx 1.47.$$

This is the largest reasonable estimate of A, and is called the <u>upper sum</u>.

We can conclude that $\frac{39}{32} \leq A \leq \frac{47}{32}$.