MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 7 | **MATHEMATICS 1000** | FALL 2023

SOLUTIONS

[4] 1. (a) We have

$$\begin{aligned}
\ln(y) &= \ln\left((x^2+3)^{x^3-4}\right) \\
&= (x^3-4)\ln(x^2+3) \\
\frac{d}{dx}[\ln(y)] &= \frac{d}{dx}[(x^3-4)\ln(x^2+3)] \\
\frac{1}{y}\cdot\frac{dy}{dx} &= 3x^2\ln(x^2+3) + \frac{1}{x^2+3}\cdot 2x\cdot(x^3-4) \\
&= 3x^2\ln(x^2+3) + \frac{2x(x^3-4)}{x^2+3} \\
\frac{dy}{dx} &= y\left[3x^2\ln(x^2+3) + \frac{2x(x^3-4)}{x^2+3}\right] \\
&= (x^2+3)^{x^3-4}\left[3x^2\ln(x^2+3) + \frac{2x(x^3-4)}{x^2+3}\right].
\end{aligned}$$

[4] (b) We have

$$\begin{aligned}
\ln(y) &= \ln\left(\frac{x\sqrt{x^5+3}}{e^x\cosh^4(x)}\right) \\
&= \ln\left(x\sqrt{x^5+3}\right) - \ln(e^x\cosh^4(x)) \\
&= \ln(x) + \ln\left(\sqrt{x^5+3}\right) - \ln(e^x) - \ln(\cosh^4(x)) \\
&= \ln(x) + \frac{1}{2}\ln(x^5+3) - x - 4\ln(\cosh(x)) \\
\frac{d}{dx}[\ln(y)] &= \frac{d}{dx}\left[\ln(x) + \frac{1}{2}\ln(x^5+3) - x - 4\ln(\cosh(x))\right] \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^5+3}\cdot(5x^4) - 1 - \frac{4}{\cosh(x)}\cdot[\sinh(x)] \\
&= \frac{1}{x} + \frac{5x^4}{2(x^5+3)} - 1 - 4\tanh(x) \\
\frac{dy}{dx} &= y\left[\frac{1}{x} + \frac{5x^4}{2(x^5+3)} - 1 - 4\tanh(x)\right] \\
&= \frac{x\sqrt{x^5+3}}{e^x\cos^4(x)}\left[\frac{1}{x} + \frac{5x^4}{2(x^5+3)} - 1 - 4\tanh(x)\right].
\end{aligned}$$

[4] 2. (a) First we have

$$f'(x) = [\arcsin(x)]' \arccos(x) + [\arccos(x)]' \arcsin(x) = \frac{\arccos(x)}{\sqrt{1-x^2}} - \frac{\arcsin(x)}{\sqrt{1-x^2}}.$$

Then

$$\begin{aligned}
f''(x) &= \frac{[\arccos(x)]'\sqrt{1-x^2} - \left[\sqrt{1-x^2}\right]' \arccos(x)}{1-x^2} \\
&\quad - \frac{[\arcsin(x)]'\sqrt{1-x^2} - \left[\sqrt{1-x^2}\right]' \arcsin(x)}{1-x^2} \\
&= \frac{-1 - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) \arccos(x)}{1-x^2} - \frac{1 - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) \arcsin(x)}{1-x^2} \\
&= \frac{-1 + \frac{x \arccos(x)}{\sqrt{1-x^2}}}{1-x^2} - \frac{1 + \frac{x \arcsin(x)}{\sqrt{1-x^2}}}{1-x^2} \\
&= \frac{-\sqrt{1-x^2} + x \arccos(x)}{(1-x^2)^{\frac{3}{2}}} - \frac{\sqrt{1-x^2} + x \arcsin(x)}{(1-x^2)^{\frac{3}{2}}} \\
&= \frac{x \arccos(x) - x \arcsin(x) - 2\sqrt{1-x^2}}{(1-x^2)^{\frac{3}{2}}}.
\end{aligned}$$

[4] (b) We begin by finding the first derivative:

$$\begin{aligned}
\frac{d}{dx}[x^3 - y^2] &= \frac{d}{dx}[4] \\
3x^2 - 2y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{3x^2}{2y}.
\end{aligned}$$

Now we differentiate again:

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[\frac{3x^2}{2y}\right] \\
&= \frac{6x(2y) - 2\frac{dy}{dx}(3x^2)}{(2y)^2} \\
&= \frac{12xy - 6x^2\frac{dy}{dx}}{4y^2} \\
&= \frac{12xy - 6x^2\left(\frac{3x^2}{2y}\right)}{4y^2} \\
&= \frac{24xy^2 - 18x^4}{8y^3} \\
&= \frac{12xy^2 - 9x^4}{4y^3}.
\end{aligned}$$

[4] 3. (a) Let y be the plane's altitude, x be its horizontal distance (along the ground) from the radar antenna, and ℓ be the straight-line distance between the plane and the radar. We are given that $y = 12$ (a constant) and $\frac{dx}{dt} = 250$. We want to find $\frac{d\ell}{dt}$ when $\ell = 13$. Since y, x and ℓ form a right triangle, the Pythagorean theorem tells us that

$$\begin{aligned}
x^2 + y^2 &= \ell^2 \\
\frac{d}{dt}[x^2 + y^2] &= \frac{d}{dt}[\ell^2] \\
2x\frac{dx}{dt} &= 2\ell\frac{d\ell}{dt} \\
x\frac{dx}{dt} &= \ell\frac{d\ell}{dt}.
\end{aligned}$$

Substituting back into the original equation, we find that when $\ell = 13$,

$$x^2 + 12^2 = 13^2 \quad \Longrightarrow \quad x^2 = 25 \quad \Longrightarrow \quad x = 5.$$

(We can ignore the possibility that $x = -5$ because of the way we have defined x.) Thus the differentiated equation becomes

$$\begin{aligned}
5 \cdot 250 &= 13\frac{d\ell}{dt} \\
\frac{d\ell}{dt} &= \frac{1250}{13},
\end{aligned}$$

that is, the straight-line distance between the radar and the plane is growing by $\frac{1250}{13}$ km/hr.

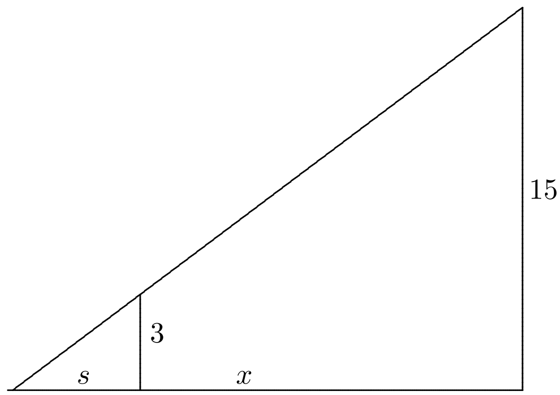

Figure 1: A girl walks away from a streetlamp.

[4] (b) Let s be the length of the girl's shadow and x be the horizontal distance between her and the streetlamp, as depicted in Figure 1. We are given that $\frac{dx}{dt} = 2$, and we want to find $\frac{ds}{dt}$ when $x = 10$.

But observe that there are two right triangles in Figure 1, and both of them contain the angle that the beam of light makes with the edge of the girl's shadow. Thus these two triangles are similar. The smaller triangle has base s and height 3. The larger triangle has base $s + x$ and height 15. Thus

$$\begin{aligned} \frac{s}{3} &= \frac{s+x}{15} \\ \frac{1}{3}s &= \frac{1}{15}s + \frac{1}{15}x \\ 4s &= x \\ 4\frac{ds}{dt} &= \frac{dx}{dt} \\ \frac{ds}{dt} &= \frac{1}{4}(2) = \frac{1}{2}. \end{aligned}$$

Hence the length of the girl's shadow is changing at a rate of $\frac{1}{2}$ foot per second. (Notice that, in fact, this rate is a constant — it isn't only true when $x = 10$.)