MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Section 4.5 **Math 1000 Worksheet** Fall 2023

SOLUTIONS

1. Let x be the length of fencing parallel to the river and y be the length of the other side of the rectangle. The quantity to be maximised is the area A. The primary equation is

$$A = xy$$

and the secondary equation is

$$x + 2y = 1000 \quad \Longrightarrow \quad x = 1000 - 2y$$

so the reduced primary equation is

$$A(y) = (1000 - 2y)y = 1000y - 2y^2.$$

This problem is defined for $y > 0$, an open interval, so we next compute

$$A'(y) = 1000 - 4y$$

and set $A'(y) = 0$, giving $y = 250$. Note that $A''(y) = -4$ so $A''(y) < 0$ for all y, and in particular for $y = 250$. Hence, by the Second Derivative Test, $y = 250$ is the absolute maximum. When $y = 250$, from the secondary equation we see that

$$x = 1000 - 2(250) = 500.$$

Thus the area is a maximum when the plot of land measures 500 metres by 250 metres.

2. Let r be the radius of the cylinder (and thus also of the hemisphere) and h be its height. The quantity to be minimised is the cost C. Note that the surface area of the cylindrical portion (including the bottom) is $\pi r^2 + 2\pi rh$ (the normal surface area of a cylinder, minus the surface area of the circle at the top) while the surface area of the hemisphere is $2\pi r^2$ (half the surface area of a sphere). Hence the primary equation is

$$C = 2(\pi r^2 + 2\pi rh) + 3.5(2\pi r^2) = 9\pi r^2 + 4\pi rh.$$

Since the volume of a cylinder is $\pi r^2 h$ and the volume of a hemisphere is $\frac{2\pi}{3}r^3$ (again, half the volume of a sphere), the secondary equation is

$$\pi r^2 h + \frac{2\pi}{3}r^3 = 1 \quad \Longrightarrow \quad h = \frac{1 - \frac{2\pi}{3}r^3}{\pi r^2}.$$

Thus the reduced primary equation is

$$C(r) = 9\pi r^2 + 4\pi r\left(\frac{1 - \frac{2\pi}{3}r^3}{\pi r^2}\right) = 9\pi r^2 + \frac{4}{r} - \frac{8\pi}{3}r^2 = \frac{19\pi}{3}r^2 + \frac{4}{r},$$

where $r > 0$. Observe that

$$C'(r) = \frac{38\pi}{3}r - \frac{4}{r^2},$$

and setting $C'(r) = 0$ yields

$$\frac{38\pi}{3}r = \frac{4}{r^2} \implies r^3 = \frac{6}{19\pi} \implies r = \sqrt[3]{\frac{6}{19\pi}}.$$

Also,

$$C''(r) = \frac{38\pi}{3} + \frac{8}{r^3} \implies C''\left(\sqrt[3]{\frac{6}{19\pi}}\right) > 0,$$

so this value of r is the absolute minimum by the Second Derivative Test. The minimum value of C, then, is

$$C\left(\sqrt[3]{\frac{6}{19\pi}}\right) = \frac{19\pi}{3}\left(\sqrt[3]{\frac{6}{19\pi}}\right)^2 + \frac{4}{\sqrt[3]{\frac{6}{19\pi}}} \approx 12.90,$$

which means that the cheapest possible cost of the tube is $12.90.

3. Let x and y be the length and height of the poster; see Figure 1.

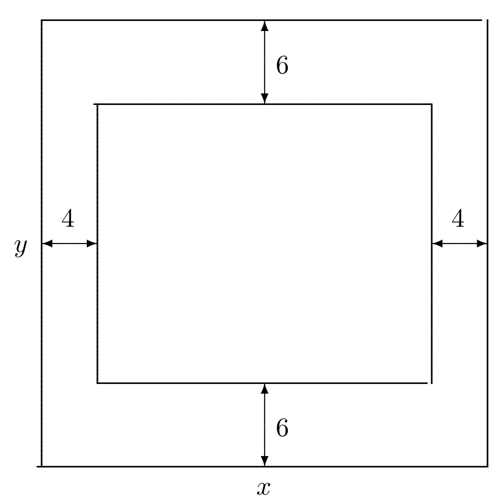

Figure 1: A poster with margins, as discussed in Question 3.

The quantity to be minimised is its area, A. The primary equation is

$$A = xy.$$

Note that the length of the printed matter, accounting for the side margins, is $x - 8$ while the height is $y - 12$. Hence the secondary equation is

$$(x-8)(y-12) = 384 \implies y = \frac{384}{x-8} + 12.$$

The reduced primary equation is

$$A(x) = x\left(\frac{384}{x-8} + 12\right) = \frac{384x}{x-8} + 12x$$

where $x > 8$ (to ensure that there is a non-negative amount of printed matter). Now

$$A' = \frac{12(x^2 - 16x - 192)}{(x-8)^2}.$$

We set $A' = 0$ and get $x = 24$ and $x = -8$, but we can disregard the latter because length must be positive. Note that

$$A'' = \frac{6144}{(x-8)^3} \quad \Longrightarrow \quad A''(24) > 0$$

so by the Second Derivative Test, $x = 24$ is an absolute minimum. By the secondary equation, when $x = 24$,

$$y = \frac{384}{24-8} + 12 = 36,$$

so the dimensions of the poster with the smallest area are 24 cm × 36 cm.

4. Let the distance between Toontown and the Roadrunner be r, and the distance between Wile E Coyote and Toontown be c. The quantity to be minimised is the distance between the Roadrunner and Wile E Coyote, ℓ, as shown in Figure 2. Note that, because he arrives in Toontown at 3:00pm after travelling at 15 km/hr, at 2:00pm Wile E Coyote must be 15 km west of Toontown.

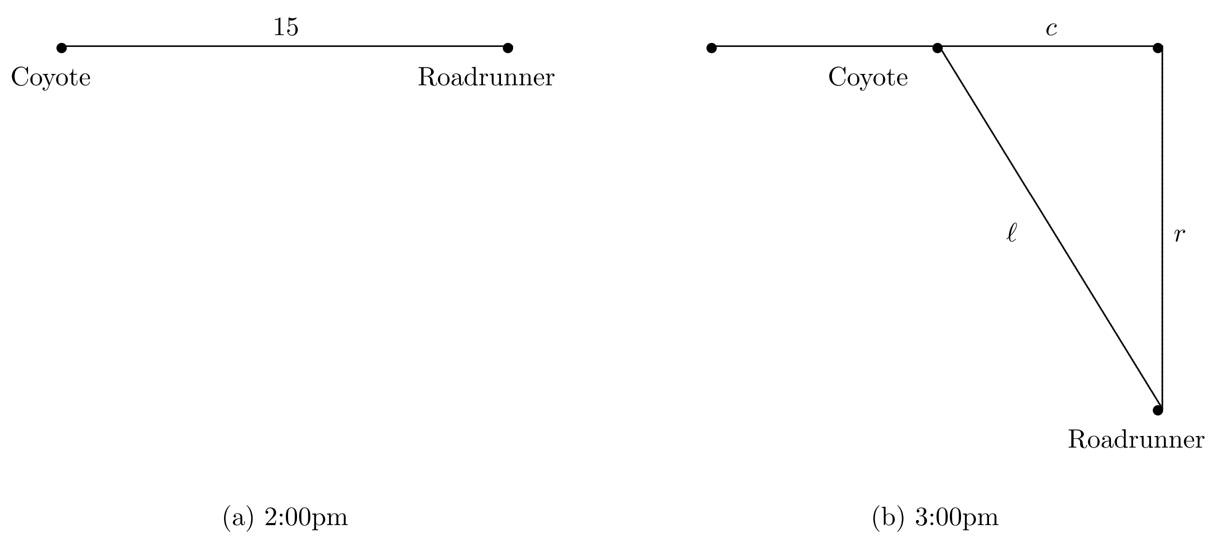

Figure 2: Wile E Coyote fails to catch the Roadrunner, as in Question 4.

The primary equation is

$$\ell = \sqrt{c^2 + r^2}.$$

To find secondary equations, let t be the time (measured in hours) elapsed since 2:00pm. Then

$$r = 20t$$

and

$$c = 15 - 15t.$$

Hence the primary equation becomes

$$\ell(t) = \sqrt{(15-15t)^2 + (20t)^2} = \sqrt{625t^2 - 450t + 225}.$$

This is defined on the closed interval $0 \leq t \leq 1$, from when Wile E Coyote launches himself towards Toon Town, to when he arrives. Observe that

$$\ell'(t) = \frac{5(50t-18)}{2\sqrt{25t^2 - 18t + 9}}.$$

If we set this equal to zero, we obtain

$$50t - 18 = 0 \quad \Longrightarrow \quad t = \frac{9}{25}.$$

Note that

$$\ell\left(\frac{9}{25}\right) = 12,$$

while at the endpoints,

$$\ell(0) = 15 \quad \text{and} \quad \ell(1) = 20.$$

Hence, by the Extreme Value Theorem, the distance between the Roadrunner and Wile E Coyote will be the smallest after $\frac{9}{25}$ hours.