

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 4.3

Math 1000 Worksheet

FALL 2023

SOLUTIONS

1. (a) First we observe that $f(x)$ is defined except when $(x - 1)^3 = 0$ so $x = 1$. Note that the numerator is non-zero, so $x = 1$ is a vertical asymptote.

To find any horizontal asymptotes, we evaluate

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2(x+2)}{(x-1)^3} &= \lim_{x \rightarrow \infty} \frac{x^3 + 2x^2}{x^3 - 3x^2 + 3x - 1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{1 + \frac{2}{x}}{1 - \frac{3}{x} + \frac{3}{x^2} - \frac{1}{x^3}} \\ &= \frac{1 + 0}{1 - 0 + 0 - 0} \\ &= 1. \end{aligned}$$

Thus $y = 1$ is a horizontal asymptote, and since $f(x)$ is a rational function, it must approach the horizontal asymptote both as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

To find any x -intercepts, we set $f(x) = 0$, so $x^2(x+2) = 0$, so $x = 0$ or $x = -2$. Thus the points $(0, 0)$ and $(-2, 0)$ are the x -intercepts. This also means that $(0, 0)$ is the y -intercept, which we could alternatively find by evaluating $f(0)$.

Now we need to find any critical points. Note that $f'(x)$ is undefined only when $x = 1$ (the vertical asymptote) so we need only consider $f'(x) = 0$, that is, $-x(5x+4) = 0$, so $x = 0$ or $x = -\frac{4}{5}$. We can now construct the sign pattern found in Figure 1. We can see that $f(x)$ is increasing on the interval $-\frac{4}{5} < x < 0$ and decreasing on the intervals $x < -\frac{4}{5}$, $0 < x < 1$ and $x > 1$. Furthermore, we have a relative minimum at $x = -\frac{4}{5}$, which is the point $(-\frac{4}{5}, -\frac{32}{243})$. We have a relative maximum at $x = 0$, which is the point $(0, 0)$.



Figure 1: Sign patterns for Question 1(a).

Finally, we find the hypercritical points. Again, $f''(x)$ is undefined only at the vertical asymptote $x = 1$. Furthermore, $f''(x) = 0$ when $2(5x+1)(x+2) = 0$, that is when $x = -\frac{1}{5}$ or $x = -2$. We therefore construct the sign pattern found in Figure 1. We

conclude that $f(x)$ is concave upward on the intervals $-2 < x < -\frac{1}{5}$ and $x > 1$ and concave downward on the intervals $x < -2$ and $-\frac{1}{5} < x < 1$. The points of inflection occur at $x = -2$ and $x = -\frac{1}{5}$, which are the points $(-2, 0)$ and $(-\frac{1}{5}, -\frac{1}{24})$. Now we can sketch the graph of $f(x)$, as found in Figure 2.

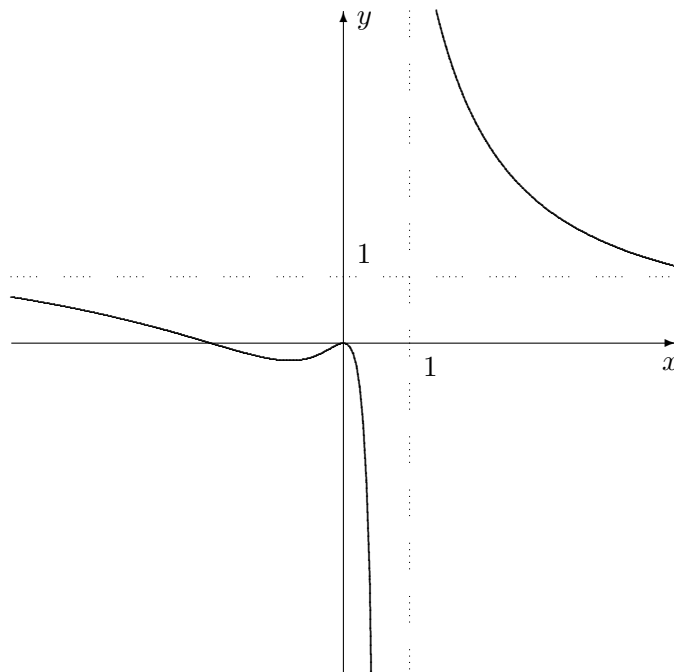


Figure 2: The graph for Question 1(a).

- (b) Observe that $f(x)$ is defined anywhere because if $x^2 + 1 = 0$ then $x^2 = -1$, which has no solutions. Thus there are no vertical asymptotes or other discontinuities. For the horizontal asymptotes, we observe that $f(x)$ is rational so we need only take one limit at infinity:

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{2x^2 - 4x + 2}{x^2 + 1} \cdot \frac{1}{x^2} = \lim_{x \rightarrow \infty} \frac{2 - \frac{4}{x} + \frac{2}{x^2}}{1 + \frac{1}{x^2}} = \frac{2 - 0 + 0}{1 + 0} = 2,$$

giving $y = 2$ as the horizontal asymptote.

Now we seek the x -intercepts. Setting $f(x) = 0$ gives

$$2x^2 - 4x + 2 = 2(x - 1)^2 = 0$$

so $x = 1$; hence $(1, 0)$ is the only x -intercept. Next we evaluate $f(0) = 2$, so $(0, 2)$ is the y -intercept.

Now we consider $f'(x)$. Observe that it never fails to exist since the denominator is always positive. Thus we set $f'(x) = 0$ so

$$4x^2 - 4 = 4(x + 1)(x - 1) = 0$$

giving $x = 1$ and $x = -1$ as the critical points. From the sign pattern given in Figure 3 we can see that $f(x)$ is increasing on $x < -1$ and $x > 1$, while it is decreasing on $-1 < x < 1$. Hence the point $(-1, 4)$ is a relative maximum while the point $(1, 0)$ is a relative minimum.



Figure 3: The sign patterns for Question 1(b).

Finally, we consider concavity. Again, $f''(x)$ never fails to exist so we set $f''(x) = 0$, giving

$$24x - 8x^3 = 8x(x^2 - 3) = 0.$$

Thus $x = 0$, $x = \sqrt{3}$ and $x = -\sqrt{3}$ are the hypercritical points. From the sign pattern, we see that $f(x)$ is concave upward on $x < -\sqrt{3}$ and $0 < x < \sqrt{3}$, and it is concave downward on $-\sqrt{3} < x < 0$ and $x > \sqrt{3}$. Hence the points $(-\sqrt{3}, 2 + \sqrt{3})$, $(0, 2)$ and $(\sqrt{3}, 2 - \sqrt{3})$ are all inflection points.

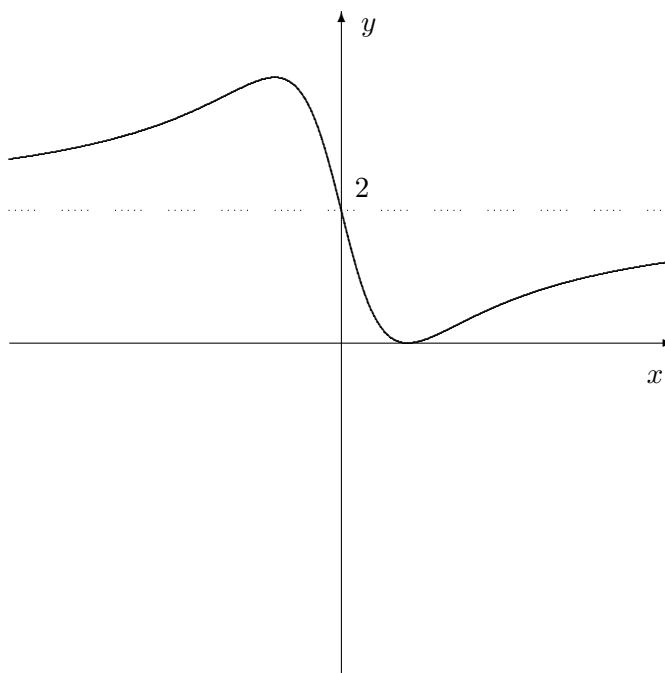


Figure 4: The graph for Question 1(b).

We can now sketch the graph depicted in Figure 4.

- (c) The domain of $f(x)$ consists of all real numbers because $x^2 + 1 > 0$ and so the square root can neither have a negative argument nor be equal to zero. Thus there are no vertical asymptotes or other discontinuities. Since this is a quasirational function, $f(x)$ may have as many as two different horizontal asymptotes, so we must check both limits at infinity. First,

$$\lim_{x \rightarrow \infty} \frac{x+3}{\sqrt{x^2+1}} \cdot \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{1 + \frac{3}{x}}{\frac{1}{\sqrt{x^2}} \sqrt{x^2+1}} = \lim_{x \rightarrow \infty} \frac{1 + \frac{3}{x}}{\sqrt{1 + \frac{1}{x^2}}} = \frac{1+0}{\sqrt{1+0}} = 1.$$

Thus the graph approaches the horizontal asymptote $y = 1$ as $x \rightarrow \infty$. Next,

$$\lim_{x \rightarrow -\infty} \frac{x+3}{\sqrt{x^2+1}} \cdot \frac{1}{x} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{3}{x}}{\frac{1}{-\sqrt{x^2}} \sqrt{x^2+1}} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{3}{x}}{-\sqrt{1 + \frac{1}{x^2}}} = \frac{1+0}{-\sqrt{1+0}} = -1.$$

This means that the graph approaches the horizontal asymptote $y = -1$ as $x \rightarrow -\infty$.

Now we check for any x -intercepts. We set $x+3=0$ so $x=-3$, and conclude that the point $(-3, 0)$ is the only x -intercept. Also, $f(0)=3$ so $(0, 3)$ is the y -intercept.

Next we consider $f'(x)$. It is never undefined, and so we set $1-3x=0$, which means that $x=\frac{1}{3}$ is the only critical point. From the sign pattern given in Figure 5 we can see that $f(x)$ is increasing on $x < \frac{1}{3}$ and decreasing on $x > \frac{1}{3}$. Hence the point $(\frac{1}{3}, \sqrt{10})$ is a relative maximum.



Figure 5: Sign patterns for Question 1(c).

Finally, we consider concavity. Again, $f''(x)$ never fails to exist so we set $f''(x)=0$, giving $3(2x+1)(x-1)=0$ and so $x=-\frac{1}{2}$ and $x=1$ are the hypercritical points. From the sign pattern, we see that $f(x)$ is concave upward on $x < -\frac{1}{2}$ and $x > 1$, and it is concave downward on $-\frac{1}{2} < x < 1$. Hence the points $(-\frac{1}{2}, \sqrt{5})$ and $(1, 2\sqrt{2})$ are both inflection points.

We can now sketch the graph depicted in Figure 6.

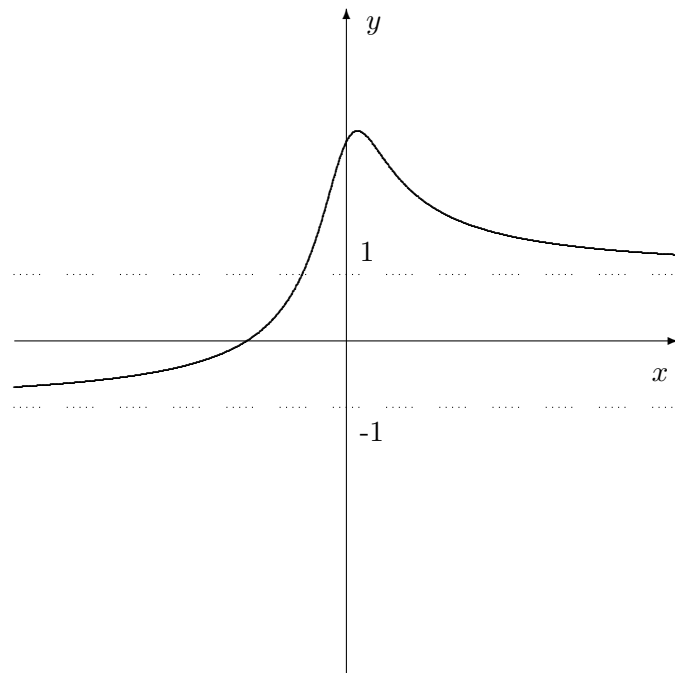


Figure 6: The graph for Question 1(c).