MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 4.2 | **Math 1000 Worksheet** | FALL 2023

SOLUTIONS

1. (a) The function is a polynomial, so it is defined for all real numbers x. Differentiation gives

$$f'(x) = 12x^3 - 24x^2 + 12x = 12x(x-1)^2,$$
$$f''(x) = 36x^2 - 48x + 12 = 12(x-1)(3x-1)$$

Setting $f'(x) = 0$ gives $x = 0$ and $x = 1$. Since $f'(x)$ is always defined, these are our critical points. Setting $f''(x) = 0$ gives $x = 1$ and $x = \frac{1}{3}$. Since $f''(x)$ is also always defined, these are the only hypercritical points. We use these values to construct the sign patterns depicted in Figure 1.

Figure 1: Sign patterns for Section 4.2, Question 1(a).

We can see that $f(x)$ is increasing on $0 < x < 1$ and $x > 1$, and decreasing on $x < 0$. There is a relative minimum at $x = 0$ but $x = 1$ is a saddle point.

The function is concave upward for $x < \frac{1}{3}$ and $x > 1$, and concave downward for $\frac{1}{3} < x < 1$. Both $x = \frac{1}{3}$ and $x = 1$ are points of inflection.

(b) Since $1 + x^2 > 0$ for all x, and a logarithmic function is defined as long as its argument is positive, the domain of $f(x)$ consists of all real numbers x. Differentiation gives

$$f'(x) = \frac{2x}{1+x^2} \quad \text{and} \quad f''(x) = \frac{-2(x-1)(x+1)}{(x^2+1)^2}.$$

Setting $f'(x) = 0$ gives $x = 0$. Since we cannot have $(x^2+1)^2 = 0$, $f'(x)$ is always defined, and so $x = 0$ is the only critical point. Setting $f''(x) = 0$ gives $x = \pm 1$. Again, $f''(x)$ is always defined, so these are the only hypercritical points. We use these values to construct the sign pattern shown in Figure 2.

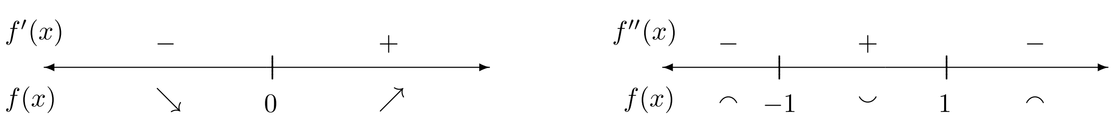

Figure 2: Sign patterns for Section 4.2, Question 1(b).

We can see that $f(x)$ is increasing on $x > 0$ and decreasing on $x < 0$. There is a relative minimum at $x = 0$ but there are no local maxima.

Furthermore, $f(x)$ is concave upward on $-1 < x < 1$ and concave downward on $x < -1$ and $x > 1$. Both $x = 1$ and $x = -1$ are inflection points.