MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 3 | **MATHEMATICS 1000-003** | FALL 2023

SOLUTIONS

[5] 1. (a) We use the Chain Rule twice:

$$\begin{aligned} y' &= 4\tan^3(e^x)\cdot[\tan(e^x)]' \\ &= 4\tan^3(e^x)\cdot\sec^2(e^x)\cdot[e^x]' \\ &= 4\tan^3(e^x)\cdot\sec^2(e^x)\cdot e^x \\ &= 4e^x\tan^3(e^x)\sec^2(e^x). \end{aligned}$$

[5] (b) We use the Product Rule, followed by the Chain Rule:

$$\begin{aligned} y' &= [x^4]'\tan(e^x) + x^4[\tan(e^x)]' \\ &= 4x^3\tan(e^x) + x^4\cdot\sec^2(e^x)\cdot[e^x]' \\ &= 4x^3\tan(e^x) + x^4\cdot\sec^2(e^x)\cdot e^x \\ &= 4x^3\tan(e^x) + x^4e^x\sec^2(e^x). \end{aligned}$$

[5] (c) We use the Chain Rule, followed by the Product Rule:

$$\begin{aligned} y' &= \sec^2(x^4e^x)\cdot[x^4e^x]' \\ &= \sec^2(x^4e^x)\cdot([x^4]'e^x + x^4[e^x]') \\ &= \sec^2(x^4e^x)\cdot(4x^3e^x + x^4e^x) \\ &= 4x^3e^x\sec^2(x^4e^x) + x^4e^x\sec^2(x^4e^x). \end{aligned}$$

[5] (d) We use the Product Rule twice:

$$\begin{aligned} f'(x) &= [x^{-7}]'7^x\sec(x) + x^{-7}[7^x\sec(x)]' \\ &= -7x^{-8}7^x\sec(x) + x^{-7}([7^x]'\sec(x) + 7^x[\sec(x)]') \\ &= -7x^{-8}7^x\sec(x) + x^{-7}(7^x\ln(7)\sec(x) + 7^x\sec(x)\tan(x)) \\ &= -7x^{-8}7^x\sec(x) + x^{-7}7^x\ln(7)\sec(x) + x^{-7}7^x\sec(x)\tan(x). \end{aligned}$$

[5] (e) We use the Quotient Rule, followed by the Chain Rule:

$$\begin{aligned}
f'(x) &= \frac{[\sin(5x)]'[\sin(5x)+1] - \sin(5x)[\sin(5x)+1]'}{[\sin(5x)+1]^2} \\
&= \frac{\cos(5x)\cdot[5x]'[\sin(5x)+1] - \sin(5x)\cos(5x)\cdot[5x]'}{[\sin(5x)+1]^2} \\
&= \frac{\cos(5x)\cdot 5\cdot[\sin(5x)+1] - \sin(5x)\cos(5x)\cdot 5}{[\sin(5x)+1]^2} \\
&= \frac{5\sin(5x)\cos(5x) + 5\cos(5x) - 5\sin(5x)\cos(5x)}{[\sin(5x)+1]^2} \\
&= \frac{5\cos(5x)}{[\sin(5x)+1]^2}.
\end{aligned}$$

[5] (f) We use the Quotient Rule, followed by the Product Rule:

$$\begin{aligned}
y' &= \frac{[x\cos(x)]'(x^2-4) - x\cos(x)[x^2-4]'}{(x^2-4)^2} \\
&= \frac{([x]'\cos(x) + x[\cos(x)]')(x^2-4) - x\cos(x)\cdot 2x}{(x^2-4)^2} \\
&= \frac{(1\cdot\cos(x) - x\sin(x))(x^2-4) - 2x^2\cos(x)}{(x^2-4)^2} \\
&= \frac{(\cos(x) - x\sin(x))(x^2-4) - 2x^2\cos(x)}{(x^2-4)^2}.
\end{aligned}$$

Optionally, we could further write

$$\begin{aligned}
y' &= \frac{x^2\cos(x) - x^3\sin(x) - 4\cos(x) + 4x\sin(x) - 2x^2\cos(x)}{(x^2-4)^2} \\
&= \frac{4x\sin(x) - x^3\sin(x) - 4\cos(x) - x^2\cos(x)}{(x^2-4)^2}.
\end{aligned}$$

[5] 2. We use implicit differentiation, applying the Product Rule to the lefthand side:

$$\begin{aligned}
\frac{d}{dx}[x^3y^3] &= \frac{d}{dx}[6x+2y] \\
\frac{d}{dx}[x^3]y^3 + x^3 \cdot \frac{d}{dx}[y^3] &= 6 + 2\frac{dy}{dx} \\
3x^2y^3 + x^3 \cdot 3y^2\frac{dy}{dx} &= 6 + 2\frac{dy}{dx} \\
3x^3y^2\frac{dy}{dx} - 2\frac{dy}{dx} &= 6 - 3x^2y^3 \\
\frac{dy}{dx}(3x^3y^2 - 2) &= 6 - 3x^2y^3 \\
\frac{dy}{dx} &= \frac{6-3x^2y^3}{3x^3y^2-2}.
\end{aligned}$$

[5] 3. We have

$$\begin{aligned}
[\sin(x)]' &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{[\sin(x)\cos(h)+\cos(x)\sin(h)]-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)}{h} + \lim_{h\to 0}\frac{\cos(x)\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \\
&= \sin(x)\cdot 0 + \cos(x)\cdot 1 \\
&= \cos(x).
\end{aligned}$$