MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 3 **MATHEMATICS 1000** Fall 2023

SOLUTIONS

[5] 1. Since this is a quasirational function, we must consider both limits at infinity. Note that the smallest power of x in the denominator is effectively x (since we treat the x^2 inside the square root as having half its actual power). First, then,

$$\begin{aligned}
\lim_{x\to\infty} \frac{5x+7}{3x-\sqrt{9x^2+2}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} &= \lim_{x\to\infty} \frac{5+\frac{7}{x}}{3-\frac{1}{x}\sqrt{9x^2+2}} \\
&= \lim_{x\to\infty} \frac{5+\frac{7}{x}}{3-\frac{1}{\sqrt{x^2}}\sqrt{9x^2+2}} \\
&= \lim_{x\to\infty} \frac{5+\frac{7}{x}}{3-\sqrt{9+\frac{2}{x^2}}} \\
&= \frac{5+0}{3-\sqrt{9-0}} \\
&= \frac{5}{0},
\end{aligned}$$

so the limit does not exist.

Next,

$$\begin{aligned}
\lim_{x\to-\infty} \frac{5x+7}{3x-\sqrt{9x^2+2}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} &= \lim_{x\to-\infty} \frac{5+\frac{7}{x}}{3-\frac{1}{x}\sqrt{9x^2+2}} \\
&= \lim_{x\to-\infty} \frac{5+\frac{7}{x}}{3-\frac{1}{-\sqrt{x^2}}\sqrt{9x^2+2}} \\
&= \lim_{x\to-\infty} \frac{5+\frac{7}{x}}{3+\sqrt{9+\frac{2}{x^2}}} \\
&= \frac{5+0}{3+\sqrt{9-0}} \\
&= \frac{5}{6}.
\end{aligned}$$

Hence this function has just one horizontal asymptote, namely $y = \frac{5}{6}$.

[5] 2. First observe that $f(3) = 3k - k + 1 = 2k + 1$. This will be defined for all k.

Next,

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2 + (k-3)x - 3k}{x^2 - (k+3)x + 3k} = \lim_{x\to 3} \frac{(x-3)(x+k)}{(x-3)(x-k)} = \lim_{x\to 3} \frac{x+k}{x-k} = \frac{3+k}{3-k}.$$

Thus the limit will exist for all $k \neq 3$.

Finally, we need $\lim_{x\to 3} f(x) = f(3)$, so we set

$$\begin{aligned} \frac{3+k}{3-k} &= 2k+1 \\ 3+k &= (2k+1)(3-k) \\ 3+k &= 6k - 2k^2 - k + 3 \\ 2k^2 - 4k &= 0 \\ 2k(k-2) &= 0, \end{aligned}$$

so $k = 0$ or $k = 2.$

[10] 3. First, we consider the points where we may obtain division by zero.

From the first definition, this occurs when

$$x^2 - 1 = (x-1)(x+1) = 0,$$

so $x = 1$ or $x = -1$. However, this definition only applies when $x < 0$, so we reject $x = 1$. When $x = -1$, direct substitution produces a $\frac{-4}{0}$ form, so this is a vertical asymptote — and therefore a non-removable discontinuity.

From the second definition, the denominator is zero when $x - 2 = 0$ so $x = 2$. Direct substitution results in a $\frac{0}{0}$ indeterminate form, so we need to take the limit:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 + 2x - 8}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+4)}{x-2} = \lim_{x\to 2}(x+4) = 6.$$

Since the limit exists, $x = 2$ is a removable discontinuity.

From the third definition, the denominator is zero when $x - 3 = 0$, so $x = 3$. However, this definition applies only when $x \geq 4$, so we may omit this result.

The other way a discontinuity might result is at the points where the definition of the function changes.

At $x = 0$, we have $f(0) = 4$. The one-sided limits are

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x-3}{x^2-1} = 3 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+2x-8}{x-2} = 4.$$

Since the one-sided limits disagree, $\lim_{x\to 0} f(x)$ does not exist, and therefore $x = 0$ is a non-removable discontinuity.

At $x = 4$, we have $f(4) = 8$. The one-sided limits are

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \frac{x^2 + 2x - 8}{x - 2} = 8 \quad \text{and} \quad \lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \frac{2x}{x - 3} = 8.$$

Since the one-sided limits are equal, $\lim_{x \to 4} f(x) = 8 = f(4)$, and hence the function is continuous at $x = 4$.

Now we can conclude that $f(x)$ possesses one removable discontinuity (at $x = 2$) and two non-removable discontinuities (at $x = -1$ and $x = 0$).