SOLUTIONS

[2]   1. (a) No, $f(x)$ is not continuous at $x = -3$; it is a removable discontinuity.

[2]   (b) No, $f(x)$ is not differentiable at $x = -3$; a function cannot be differentiable at a point if it is not continuous at that point.

[2]   (c) No, $f(x)$ is not continuous at $x = 0$; it is a non-removable discontinuity.

[1]   (d) No, $f(x)$ is not differentiable at $x = 0$; a function cannot be differentiable at a point if it is not continuous at that point.

[1]   (e) Yes, $f(x)$ is continuous at $x = 2$.

[1]   (f) Yes, $f(x)$ is differentiable at $x = 2$.

[1]   (g) Yes, $f(x)$ is continuous at $x = 4$.

[2]   (h) No, $f(x)$ is not differentiable at $x = 4$; there is a cusp (or a "sharp corner") at this point.

[5]   2. Observe that $f(x)$ is a rational function, so we need only consider one of the limits at infinity. Then

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^3(5 - 6x)}{(3x^2 + 1)^2}$$

$$= \lim_{x \to \infty} \frac{5x^3 - 6x^4}{9x^4 + 6x^2 + 1} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}}$$

$$= \lim_{x \to \infty} \frac{\frac{5}{x} - 6}{9 + \frac{6}{x^2} + \frac{1}{x^4}}$$

$$= \frac{0 - 6}{9 + 0 + 0}$$

$$= \frac{-6}{9}$$

$$= -\frac{2}{3}.$$

Hence the only horizontal asymptote is the line $\boxed{y = -\frac{2}{3}.}$

[7] 3. (a) We have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2 - 4(x+h) + 7] - [x^2 - 4x + 7]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 4x - 4h + 7 - x^2 + 4x - 7}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 4h}{h}$$

$$= \lim_{h \to 0} (2x + h - 4)$$

$$= 2x - 4.$$

[3]   (b) From part (a), $m = f'(1) = -2$. Furthermore, $y = f(1) = 4$. Thus the equation of the tangent line is

$$y - 4 = -2(x - 1) \quad \implies \quad y = -2x + 6.$$

[5] 4. (a) First, $f(0) = 3 \cdot 0 + 8 = 8$, which is defined. Next we must check the one-sided limits to determine if $\lim_{x \to 0} f(x)$ exists. We have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x+2}{x-1} = \frac{2}{-1} = -2$$

and

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x^2 - 4x + 4}{x - 2} = \frac{4}{-2} = -2.$$

Since the one-sided limits agree,

$$\lim_{x \to 0} f(x) = -2$$

and so the limit exists. However, $f(0) \neq \lim_{x \to 0} f(x)$ so $f(x)$ is discontinuous at $x = 0$.

Because the limit exists, the discontinuity is removable.

[8]   (b) Since part (a) has explored the only point at which the definition of $f(x)$ changes, we must consider any points which would make each part of the piecewise definition undefined.

First, then, we set $x^2 - 1 = 0$ so $x^2 = 1$ and $x = \pm 1$. However, because the first definition applies only for $x < 0$, we can neglect $x = 1$. Because $f(-1) = \frac{1}{0}$, we know that $\lim_{x \to 1} f(x)$ does not exist and hence $x = -1$ is a non-removable discontinuity.

The second part of the definition, $f(x) = 3x + 8$, is a polynomial and hence is always defined.

Finally, we set $x - 2 = 0$ so $x = 2$. Because $f(2) = \frac{0}{0}$, we use the Cancellation Method:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4x + 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)^2}{x - 2} = \lim_{x \to 2} (x - 2) = 0.$$

Since the limit exists, $x = 2$ is a removable discontinuity.